Eigenvalue problems:

Suppose we want to solve $Ax + \varepsilon B(x) = \lambda x$, $\varepsilon \ll 1$.

Expand: $x = x_0 + \varepsilon x_1 + \cdots$
$\lambda = \lambda_0 + \varepsilon \lambda_1 + \cdots$

$$\Rightarrow \begin{cases} (A - \lambda_0) x_0 = 0 \\ (A - \lambda_0) x_1 = \lambda_1 x_0 - B(x_0) \end{cases}$$

$\Rightarrow$ λ_0, x_0 is eigenvalue / eigenfunction of A;

Suppose that λ_0 is a simple eigenvalue of A;

then $\exists\ x_0^*$ s.t. $x_0^*(A - \lambda_0) = 0$.

Remark: if A is symmetric then $x_0^* = x_0^t$.

So we then obtain: $x_0^*(\lambda_1 x_0 - B(x_0)) = 0$

$$\Rightarrow \boxed{\lambda_1 = \frac{x_0^* B(x_0)}{x_0^* x_0}}$$

Now if A is $N \times N$ then $A - \lambda_0 I$ has rank $N-1$; so that x_1 is determined up to an arbitrary constant; since $x_0 \in NS(A - \lambda_0 I)$, we can write $x_1 = \hat{x}_1 + C x_0$ where $\hat{x}_1$ is uniquely determined by imposing the condition $x_0^* \hat{x}_1 = 0$.

Example: Consider $\begin{cases} y'' + \lambda y - \varepsilon y^3 = 0 \\ y(0) = 0 = y(\pi) \end{cases}$, $\varepsilon \ll 0$.

Expand: $y = y_0 + \varepsilon y_1 + \cdots$; $\lambda = \lambda_0 + \varepsilon \lambda_1 + \cdots$

$$y_0 = A\sin(nx) \; ; \quad \lambda_0 = n^2 \; ; \quad y_1(0) = 0 = y_1(\pi)$$

$$y_1'' + \lambda_0 y_1 = + y_0^3 - \lambda_1 y_0$$

Now: $\displaystyle\int_0^\pi y_1'' y_0 = \cancel{y_1' y_0 \Big|_0^\pi} - \cancel{y_1 y_0' \Big|_0^\pi} + \int_0^\pi y_1 y_0''$

$$\Rightarrow \int_0^\pi (y_1'' + \lambda_0 y_1) y_0 = \int_0^\pi y_1 (y_0'' + \lambda_0 y_0) = 0$$

$$\Rightarrow \lambda_1 = + \frac{\int_0^\pi y_0^4}{\int_0^\pi y_0^2} = + A^2 \frac{3}{4}$$

To determine y_1, we decompose

$$+ y_0^3 - \lambda_1 y_0 = \sum_{n=1}^\infty a_n \sin(nx) \; ;$$

$$y_0^3 = A^3 \sin^3(xn) = A^3\left(\frac{3}{4}\sin(xn) - \frac{1}{4}\sin 3(xn)\right)$$

and moreover, $\displaystyle\int_0^\pi y_0 (+y_0^3 - \lambda_1 y_0) = 0$

$$\Rightarrow a_n = 0$$

$$\Rightarrow y_1'' + \lambda_0 y_1 = \frac{-A^3}{4} \sin(3xn)$$

So <u>set</u>: $y_1 = B \sin(3nx)$;

$$\Rightarrow \quad B(-8n^2) = \frac{-A^3}{4} \quad \Rightarrow B = +\frac{A^3}{32n^2}$$

$$\boxed{\begin{aligned} y &\sim A\sin(nx) + \frac{A^3\varepsilon}{32n^2}\sin(3nx) \\ \lambda &\sim n^2 + A^2\frac{3}{4}\varepsilon \end{aligned}}$$

Bifurcation from constant state

Consider $\begin{cases} y'' + \lambda y - y^3 = 0, & x \in (0, \pi) \\ y(0) = 0 = y(\pi) \end{cases}$

We are interested in small solutions $y \ll 1$

Then $y^3 \ll y$ so that to leading order, $y'' + \lambda y \sim 0$

- Thus if $\lambda < 1$ then the only small sol'n is $y = 0$
- As λ crosses 1, another small sol'n appears,

  $y \sim \varepsilon \sin x, \quad \varepsilon \ll 1$
- This non-constant solution bifurcates from $y = 0$
- Question: How is ε related to λ?

Expand: $\lambda = 1 + \varepsilon^p \lambda_1 \; ; \quad y = \varepsilon(y_0 + \varepsilon^q y_1)$

where p, q are to be determined;

$$y_0'' + \varepsilon^q y_1'' + (1 + \varepsilon^p \lambda_1)(y_0 + \varepsilon^q y_1) - \varepsilon^2 (y_0 + \varepsilon^q y_1)^3$$

$\Rightarrow$ set $\boxed{p = q = 2}$

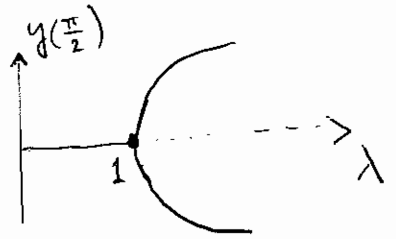
$$y_0'' + y_0 = 0 \quad \rightarrow \quad y_0 = \sin x$$

$$y_1'' + y_1 = + y_0^3 - \lambda_1 y_0$$

$$\Rightarrow \quad \int y_0^4 + \lambda_1 \int y_0^2 = 0 \Rightarrow \boxed{\lambda_1 = + \frac{3}{4}}$$

$$\Rightarrow \quad y \sim \pm \varepsilon \sin x, \quad \lambda = 1 + \varepsilon^2 \frac{3}{4}$$

[pitchfork bifurcation]



Double / multiple eigenvalues:

Suppose A is real, symmetric and has an eigenvalue λ_0 with multiple eigenvectors $v_1, \dots, v_K$. Now consider the perturbed pbm:

$$Av + \varepsilon B(v) = \lambda v .$$

Expand: $v = \sum_{i=1}^{K} a_i v_i + \varepsilon w$

$\lambda = \lambda_0 + \varepsilon \lambda_1$;

Now suppose B is linear and $v_1, \dots v_K$ are orthonormal, i.e. $(v_i, v_j) = \delta_{ij}$.

We get: $(A - \lambda_0 I) w = - B a_i v_i + \lambda_1 a_i v_i$

$$\Rightarrow \quad \lambda_1 a_i \underbrace{(v_i, v_j)}_{\delta_{ij}} = (v_j , B v_i a_i)$$

$\Rightarrow$ λ_1 is an eigenvalue of a $K \times K$ pbm:

$$M \vec{a} = \lambda_1 \vec{a} \quad \text{where}$$

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_K \end{pmatrix} , \quad M_{ij} = (v_j \, B v_i) .$$

Example:
$$\begin{cases} \Delta u + \lambda u = 0 & \text{on} \quad D = [0,\pi]^2 \\ u = 0 & \text{on} \quad \partial D \end{cases}$$

This problem has a double eigenvalue $\lambda = 5$ corresponding to

$$u = V_0 = (\sin x \ \sin 2y)\, \tfrac{2}{\pi}$$
$$u = V_1 = (\sin 2x \ \sin y)\, \tfrac{2}{\pi}.$$

Note that $(V_0, V_1) = 0$; $(V_0, V_0) = 1$
$(V_1, V_1) = 1$.

Now consider a perturbed plm

$$\Delta u + (\lambda + \varepsilon a(x,y))\, u = 0$$

Expand: $\lambda = \lambda_0 + \varepsilon \lambda_1$, $u = u_0 + \varepsilon u_1$

Near $\lambda_0 = 5$: we expand as:

$$u = c_0 V_0 + c_1 V_1 + \varepsilon w$$

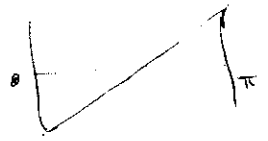
$$\Delta w + \lambda_0 w = -a u_0 - \lambda_1 u_0$$

$$\Rightarrow \left(V_i, (a u_0 + \lambda_1 u_0)\right) = 0, \quad i = 0, 1$$

$$\Rightarrow \begin{cases} c_0 \overbrace{(V_0 V_0)}^{1} \lambda_1 = -(a, V_0^2) c_0 - (a, V_0 V_1) c_1 \\ c_1 \underbrace{(V_1 V_1)}_{1} \lambda_1 = -(a V_0 V_1) c_0 - (a V_1 V_1) c_1 \end{cases}$$

$$\Rightarrow \lambda_1 \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} -(a V_0^2) & -(a V_0 V_1) \\ -(a V_0 V_1) & -(a V_1^2) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

Ex: Take $a(x,y) = \begin{pmatrix} x - \frac{\pi}{2} \\ y - \frac{\pi}{2} \end{pmatrix}$;



then

$$(a, V_0^2) = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi (\sin x \sin 2y)^2 \begin{pmatrix} x - \frac{\pi}{2} \\ y - \frac{\pi}{2} \end{pmatrix} dx\, dy =$$

$$= 0$$

$$(a, V_0 V_1) = \frac{4}{\pi^2}\int (\sin x \sin 2x)\left(x - \frac{\pi}{2}\right) \int_0^\pi (\sin y)(\sin 2y) \begin{pmatrix} \\ y - \frac{\pi}{2} \end{pmatrix} dy$$

$$= \frac{4}{\pi^2}\left(-\frac{8}{9}\right)^2 = \alpha$$

$$(a, V_1^2) = 0$$

$$\Rightarrow \lambda_1 \text{ is an eigenvalue of } \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = \pm\sqrt{\alpha}$$

$$\boxed{\lambda_1 = \pm \frac{16}{9}\frac{1}{\pi}}$$

Domain perturbations

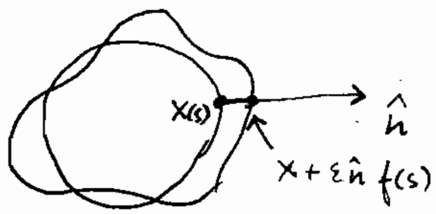
Let D_0 be a domain in $\mathbb{R}^2$ whose boundary ∂D is parametrized by $\partial D = \{ x(s) : s = a \dots b \}$

Let D_ε be a domain with the boundary given by

$$\partial D_\varepsilon = \{ x(s) + \hat{n} f(s) \varepsilon : s = a \dots b \}$$

where $\hat{n}$ is the normal at $x(s) \in \partial D_0$
and $f(s) : [a, b] \to \mathbb{R}$ is a smooth fcn:

Suppose that (u_0, λ_0) solve:



$$\begin{cases} \Delta u_0 + \lambda_0 u_0 = 0, & x \in D_0 \\ u_0 = 0, & x \in \partial D_0 \end{cases}$$

<u>Pbm</u>: Find sol'n to the perturbed pbm:

$$\begin{cases} \Delta u + \lambda u = 0, & x \in D_\varepsilon \\ u = 0, & x \in \partial D_\varepsilon . \end{cases}$$

<u>Sol'n</u>: We expand $u = u_0 + \varepsilon u_1 + \dots$, $\lambda = \lambda_0 + \varepsilon \lambda_1 + \dots$

to get $\quad \Delta u_1 + \lambda_0 u_1 = -\lambda_1 u_0$

Integrating by parts, note that for any u_0, u_1 we have:

$$\int_{D_0} u_0 \Delta u_1 = \int_{\partial D_0} u_0 \nabla u_1 \cdot \hat{n} \, dS(x) - \int_{\partial D_0} u_1 \nabla u_0 \cdot \hat{n} \, dS(x) + \int_{D_0} u_1 \Delta u_0$$

[Green's second identity].

Now $u_0 = 0$ on ∂D_0 and for $x = x_0 + \hat{n}\,\varepsilon f(s)$

we compute:

$$u_0(x) = \underbrace{\cancel{u_0(x_0)}}_{0} + \varepsilon f(s)\ \hat{n}\cdot\nabla u_0(x_0)$$

and $u(x) = u_0(x) + \varepsilon\, u_1(x) = 0$ for $x \in \partial D_\varepsilon$

$$\Rightarrow \varepsilon\Big(f(s)\ \hat{n}\,\nabla u_0(x_0) + u_1(x)\Big) = 0$$

$$\Rightarrow\quad u_1(x_0) = u_1(x) + O(\varepsilon) = -f(s)\ \hat{n}\cdot\nabla u_0(x_0)$$

So we obtain: $\displaystyle\int_{D_0} u_0\,\Delta u_1 = \int_{\partial D_0} f(s)\,(\hat{n}\cdot\nabla u_0)^2\, dS(x) + \int_{D_0} u_1\,\Delta u_0$

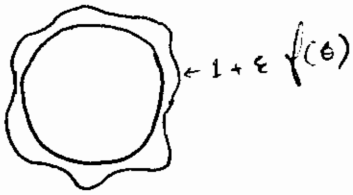
$$\Rightarrow\quad \int_{\partial D_0} f(s)\,(\hat{n}\cdot\nabla u_0)^2\, dS(x) = -\int_{D_0} \lambda_1\, u_0^2\, dx$$

$$\Rightarrow\quad \boxed{\lambda_1 = -\frac{\int_{\partial D_0} f(s)\,(\hat{n}\cdot\nabla u_0)^2\, dS(x)}{\int_{D_0} u_0^2\, dx}}$$

<u>Example of domain perturbation:</u>

Let $D_\varepsilon = \{ (x,y) : \sqrt{x^2+y^2} < 1 + \varepsilon f(\theta) \}$
where $\theta = \arctan \frac{y}{x}$

$D_0 =$ unit disk;



$$\begin{cases} \Delta u + \lambda u = 0 \ \text{on} \ D_\varepsilon \\ u = 0 \quad \text{on} \ \partial D_\varepsilon \end{cases}$$

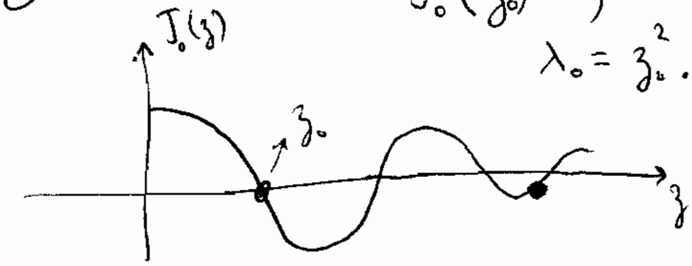
Principal eigenvalue given by

$$u_0 = J_0(\sqrt{\lambda_0}\, r) \quad , \quad r = \sqrt{x^2+y^2}$$

where J_0 is a Bessel function and $\sqrt{\lambda_0}$ its first root,

$$\begin{cases} J_{0_{zz}} + \frac{1}{z} J_{0_z} - J_0 = 0 \\ J_0'(0) = 0 \end{cases}$$

$J_0(z_0) = 0,$
$\lambda_0 = z_0^2.$



$$\Rightarrow \quad \int_D u_0^2 \, \cancel{dx} = 2\pi \int_0^1 r\, J_0^2(\sqrt{\lambda}\, r)\, dr \ ; \quad \int_{\partial D} f(s) (\partial_n u_0)^2$$

$$= \left(\int_0^{2\pi} f(\theta)\, d\theta \right) \lambda_0 J_0'(\sqrt{\lambda_0})$$

<u>Homework:</u> $\int_0^1 r\, J_0^2(\sqrt{\lambda_0}\, r)\, dr = \frac{1}{2} \left(J_0'(\sqrt{\lambda_0}) \right)^2$

$$\Rightarrow \quad \lambda_1 = -2\lambda_0 \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta = -2\lambda_0 \overline{f}$$

[in particular, $\lambda_1 = 0$ if avg $f = 0$]