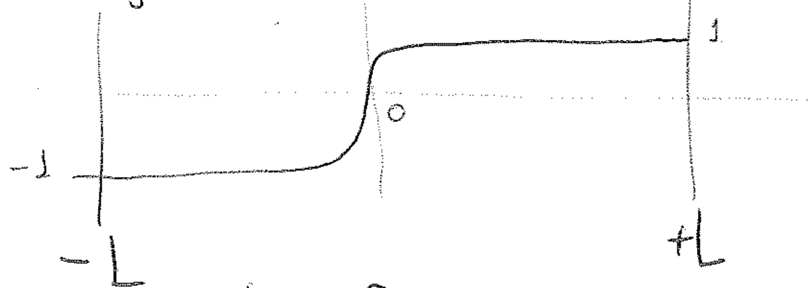
Interface stability:

$$\begin{cases} u_t = u_{yy} + f(u); & f(u) = 2u - 2u^3 \\ u_y(\pm L, t) = 0 \end{cases}$$

- Steady state: $U(y) \sim U_0(y) = \tanh y$



- What about stability??
  - Linearize: $u(y,t) = U(y) + e^{\lambda t}\varphi(y)$

$$\begin{cases} \lambda\varphi = \varphi_{yy} + f'(U)\varphi \\ \varphi_y(\pm L) = 0 \end{cases}$$

- To leading order, $\varphi \sim U_{0y}$ and $\lambda \sim 0$
- Solvability condition to determine correction to λ:

$$\lambda \int_{-L}^{L} \varphi U_y = \int_{-L}^{L} U_y\left(\varphi_{yy} + f'(U)\varphi\right)$$

$$= \left(U_y \varphi_y - U_{yy}\varphi\right)\Big|_{-L}^{L} + \int_{-L}^{L} \varphi\underbrace{\left(U_{yyy} + f'(U)U_y\right)}_{0}$$

$$\Rightarrow \lambda \int_{-L}^{L} U_{0y}^2 = -2U_{yy}(L)\varphi(L) \quad \left[\text{since } U_{yy} \text{ odd}, \ \varphi \text{ even}\right]$$

$$u_{yy} + f(u) = 0, \quad u_y(\pm L) = 0, \quad L \gg 1$$

$f(u) = 2u - 2u^3$

<u>Leading order:</u> $u \sim u_0(y) = \tanh(y)$

<u>Large y:</u> $u_0 \sim 1 - 2e^{-2y}$

$$u_{0y} \sim 4e^{-2y} \ll 1$$

- Note that $u \sim u_0$ <u>almost</u> satisfies $u_y(L) = 0$ but it misses by $O(e^{-2L})$, exponentially small amount.

- Near $y \sim L$, <u>expand</u> $u = u_0 + R$ where R is the remainder; $R \to 0$ when $L - y \gg 1$

<u>Then</u> $R_{yy} + \underbrace{f'(u_0)}_{(2-6u_0^2) = -4} R \sim 0$; $u_0 \sim 1$

$$R_{yy} - 4R \sim 0 \Rightarrow R \sim Ae^{2(y\ \ )} + Be^{-2(y\ \ )}$$

Moreover, $R \to 0$ as $y \to 0^+ \Rightarrow \boxed{B = 0}$

$$\Rightarrow u = 1 - 2e^{-2y} + Ae^{2y}$$

<u>Moreover:</u> $u_y(L) = 0 \Rightarrow 4e^{-2L} + 2Ae^{2L} = 0$

$$\Rightarrow \boxed{A = -2e^{-4L}}$$

$$\Rightarrow \boxed{u(y) \sim 1 - 2e^{-2y} - 2e^{-4L}e^{2y}}$$

$$\Rightarrow \boxed{u_{yy}(L) \sim -16e^{-2L}}$$

To determine $\varphi(L)$:

Near $y \sim L$, $u_0 \sim 1$, $\varphi_{yy} - 4\varphi \sim 0$

$\Rightarrow \varphi \sim A e^{2y} + B e^{-2y}$

- $\varphi \sim u_{0y} \sim 4 e^{-2y}$ for $1 \ll y \ll L$

$\Rightarrow \boxed{B = 4}$

- $\varphi'(L) = 0 \Rightarrow \boxed{A = 4 e^{-4L}}$

$\varphi \sim 4 e^{-4L} e^{2y} + 4 e^{-2y}$ for $\begin{cases} y < L, \\ y \sim L \end{cases}$

$\Rightarrow \boxed{\varphi(L) = 8 e^{-2L}}$

We previously found: $\boxed{u_{yy}(L) = -16 e^{-2L}}$

Also: $\int_{-L}^{L} u_{0y}^2 \sim \int_{-\infty}^{\infty} \operatorname{sech}^4 y \, dy = \frac{4}{3}$

$\Rightarrow \frac{4}{3}\lambda \sim 256 e^{-4L}$

$\boxed{\lambda \sim 192 e^{-4L}}$ $\boxed{\lambda > 0}$

$\Rightarrow$ Interface is unstable !!

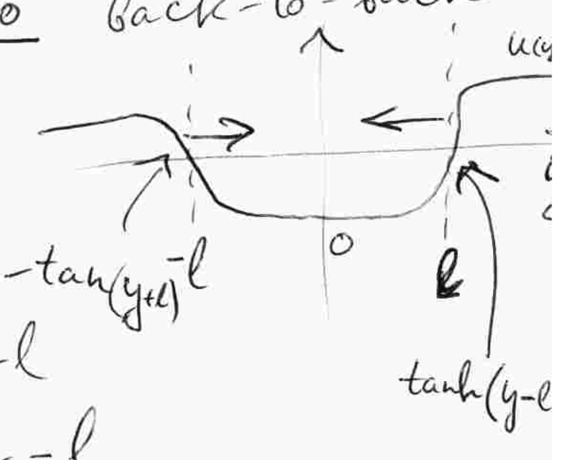
Comparison with numerics:

Take $L = 3.0 \Rightarrow$ λ_{numer} 0.00118009, $\lambda_{asympt} = 0.00117968$ $\Big\}$ 0.34 % relative error!

Interface dynamics:

Consider $u_t = u_{yy} + f(u)$, $f(u) = 2u - 2u^3$ on all of $y \in \mathbb{R}$ but suppose that the solution consists of <u>two</u> back-to-back interfaces:

$u(y,t) \sim$



That is:

$$u \sim \begin{cases} \tanh(y-l) & \text{for } y \sim l \\ -\tanh(y+l) & \text{for } y \sim -l \\ 1 & \text{for } |y| > l \\ -1 & \text{for } |y| < l \end{cases}$$

This is equivalent to the problem on half-space

$$(*) \quad \begin{cases} u_t = u_{yy} + f(u), & y > 0 \\ u_y(0,t) = 0 \end{cases}$$

We seek sol'n to $(*)$ of the form

$$u(y,t) \sim u_0(y - l) \quad \text{where } \boxed{l = l(t)}$$

is the <u>position of the interface that changes with time</u>.

Assume that $l \gg 1$ and $l'(t)$ is very small [slow dynamics]; also $u_0(y-l) = \tanh(y-l)$

Set $u(y,t) = u_0(y-l) + R(y)$

where $R \ll 1$ is a remainder.

Near $y \sim 0$, we have: $y - l \ll -1$

$$\Rightarrow \tanh(y-l) \sim -1 + 2e^{2(y-l)}$$

~~so that~~ and R satisfies:

$$R_{yy} - 4R \approx 0 \Rightarrow R = Ae^{-2y}$$

$$u \sim -1 + 2e^{-2l}e^{2y} + Ae^{-2y}$$

$$u_y(0) = 0 \Rightarrow \boxed{A = 2e^{-2l}}$$

Solvability:

$$\int_0^{\infty}\left(-l_t u_y = R_{yy} + f'(u_0)R\right)u_{0y}\,dy$$

$$\Rightarrow -l_t \frac{4}{3} = \left(R_y u_{0y} - R u_{0yy}\right)\Big|_0^{\infty}$$

$$\Rightarrow l_t = \frac{3}{4}\left(R_y u_{0y} - R u_{0yy}\right)\Big|_{y=0}$$

At $y=0$:

$$R = 2e^{-2l}e^{-2y} \qquad u_0 = -1 + 2e^{-2l}e^{2y}$$

$$R_y = -4e^{-2l} \qquad u_{0y} = 4e^{-2l}$$

$$u_{0yy} = 8e^{-2l}$$

$$\Rightarrow R_y u_{0y} - R u_{0yy}\Big|_{y=0} = -32e^{-4l}$$

$$\Rightarrow \boxed{\frac{d}{dt} l(t) = -24\, e^{-4l}} \qquad (*)$$

Summary: $u(y,t) \sim \tanh(y - l(t))$

where $l(t)$ satisfies $(*)$

~~is the~~ describes the motion of the interface for the system

$$\begin{cases} u_t = u_{yy} + 2u - 2u^3, & y \geq 0 \\ u_y(0,t) = 0 \end{cases}$$

- Note that $\frac{d}{dt} l < 0$

  $\Rightarrow$ interface moves to the left until it merges with the left boundary.

- Explicit sol'n to $(*)$ is:

$$l = \frac{1}{4} \ln\left(e^{4l_0} - 96t\right)$$

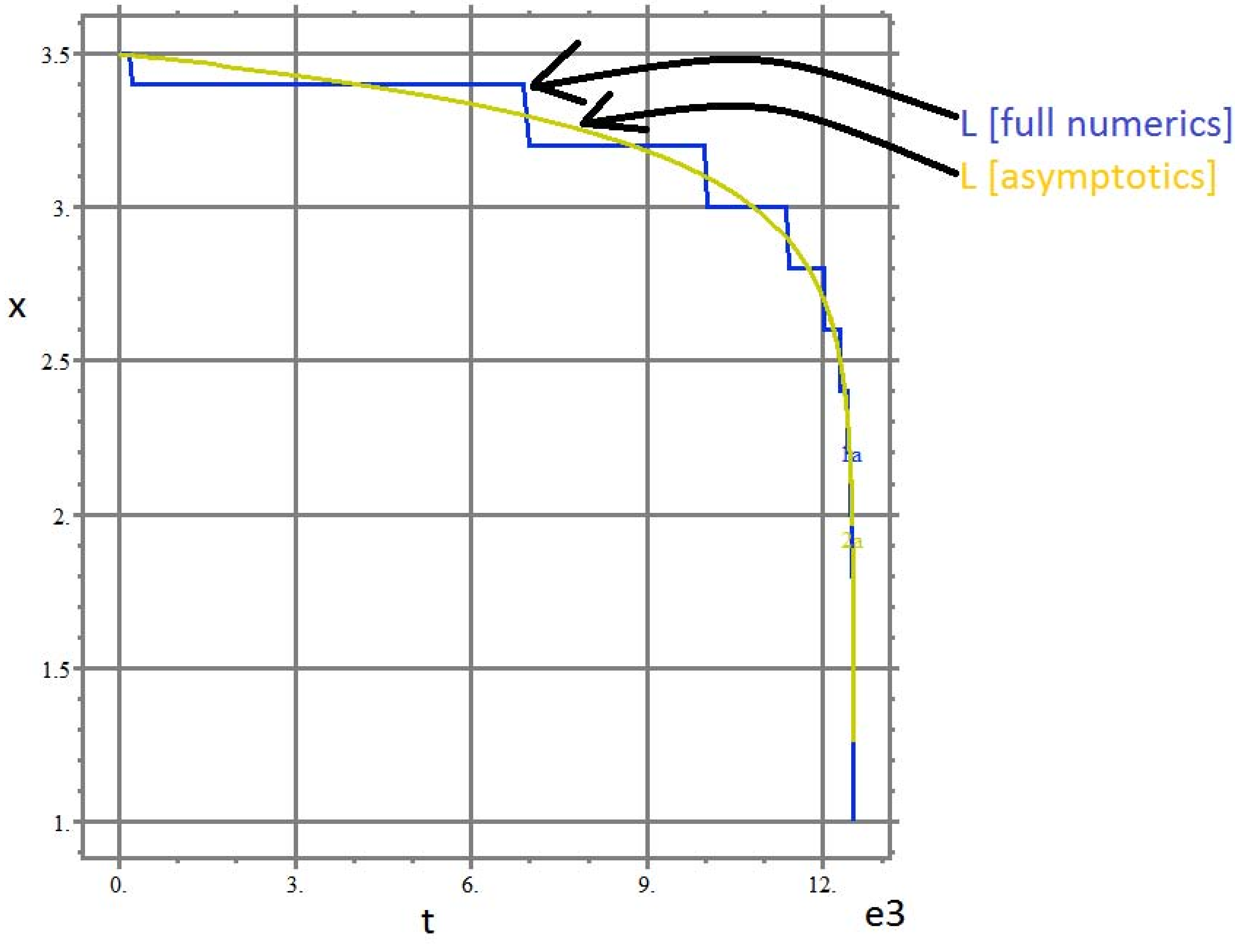
where $l_0 = l(0)$.

- Collapse occurs at $\boxed{t_{\text{collapse}} \sim \frac{1}{96} e^{4l_0}}$

- Eg: $l_0 = 3.5 \Rightarrow t_{\text{collapse}} \sim 12527.13$

  – see next page for comparison with full numerics

Example: l_0=3.5, => $t_{collapse}$=12527.13



(see the flexPDE script front1d-two.pde)