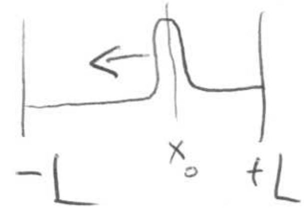
Spike motion

$$(GM)\quad \begin{cases} u_t = \varepsilon^2 u_{xx} - u + \dfrac{u^2}{v} & \text{on } [-L, L] \\ 0 = v_{xx} - v + \dfrac{u^2}{\varepsilon} \end{cases}$$

- Consider a <u>single spike</u> located at $x_0 \in [-L, L]$
- It will slowly move towards the center.



- <u>Q</u>: What are the eq'n of motion?

<u>Inner region</u>: Let $x = x_0 + \varepsilon y$, $u(x) = U(y)$, $v(x) = V(y)$

$$\Rightarrow \quad U_t = U_{yy} - U + \frac{U^2}{V}, \quad V_{yy} + \varepsilon U^2 + O(\varepsilon^2) = 0$$

<u>Anzatz</u>: $x_0 = x_0(\varepsilon^2 t)$; $u(x,t) = U(y) = U\left(\dfrac{x - x_0(\varepsilon^2 t)}{\varepsilon}\right)$

<u>Then</u> $$-\varepsilon\, x_0'(\varepsilon^2 t)\, U_y = U_{yy} - U + \frac{U^2}{V}$$

<u>Expand</u>: $U = U_0 + \varepsilon U_1 + \cdots$
$V = V_0 + \varepsilon V_1 + \cdots$

$$\Rightarrow \quad U_{0yy} - U_0 + \frac{U_0^2}{V_0} = 0, \quad \begin{cases} V_0(y) = V_0, \text{ const.} \\ U_0 = V_0\, w(y), \quad w_{yy} = w - w^2 \end{cases}$$

$$(*) \quad \cancel{V} - x_0'\, U_{0y} = U_{1yy} - U_1 + 2U_1 \frac{U_0}{V_0} - \frac{U_0^2}{V_0^2} V_1$$

Multiply (*) by U_{0y} and integrate, $y \in [-\infty, \infty]$ to get:

$$-x_0' \int_{-\infty}^{\infty} U_{0y}^2 \, dy = -\frac{1}{V_0^2} \int_{-\infty}^{\infty} U_0^2 U_{0y} V_1 \, dy$$

$$= +\frac{1}{3V_0^2} \int U_0^3 V_{1y} \, dy$$

Now $\quad V_{1yy} = U_0^2 \Rightarrow V_{1y} = \int_0^y U_0^2 + A$

where A is to be determined.

Outer region: $\begin{cases} V_{xx} - V = 0, & x \neq x_0 \\ V_x(x_0^+) - V_x(x_0^-) + \int_{-\infty}^{\infty} U_0^2 \, dy = 0 \end{cases}$

We obtain: $\quad V = \left(\int_{-\infty}^{\infty} U_0^2 \, dy \right) G(x, x_0)$

where $G(x, x_0)$ satisfies:

$$\begin{cases} G_{xx} - G = 0, & x \neq x_0 \\ G_x(x_0^+, x_0) - G_x(x_0^-, x_0) = -1, & \\ G_x(\pm L, x_0) = 0, & G \text{ cont at } x = x_0 \end{cases}$$

I.E. $\begin{cases} G_{xx} - G = -\delta(x - x_0) \\ G_x = 0 \text{ at } x = \pm L \end{cases}$

Then G has sol'n:

$$G(x,x_0) = B \begin{cases} \cosh(x-L)\cosh(x_0+L), & x>x_0 \\ \cosh(x_0-L)\cosh(x+L), & x>x_0 \end{cases}$$

$$B\Big(\underbrace{\sinh(x_0-L)\cosh(x_0+L) - \sinh(x_0+L)\cosh(x_0-L)}_{-\sinh(2L)} \Big) = -1$$

$$\Rightarrow \quad B = \frac{1}{\sinh(2L)} \qquad [\text{indep. of } x_0]$$

<u>Set</u> $x = x_0 + \varepsilon y$. If $x > x_0$ then

$$\begin{aligned} V &= \int u_0^2 \, G(x+\varepsilon y, x_0) \\ &= \int u_0^2 \, G(x_0,x_0) + \int u_0^2 \, \varepsilon y \, G_x(x_0^+, x_0) + O(\varepsilon^2) \\ &= V_0 + V_1(y)\varepsilon, \quad y \gg 1 \end{aligned}$$

$$\Rightarrow \begin{cases} V_0 = \left(\int u_0^2\right) G(x_0,x_0) \\ V_1 \sim y\left(\int_{-\infty}^{\infty} u_0^2\right) G_x(x_0^{\pm}, x_0), & \text{as } y \to \pm\infty \end{cases}$$

<u>So we get:</u>

$$\begin{cases} \left(\int_0^{\infty} u_0^2\right) + A = \int_{-\infty}^{\infty} u_0^2 \, G_x(x_0^+, x_0) \\ \int_0^{-\infty} u_0^2 + A = \int_{-\infty}^{\infty} u_0^2 \, G_x(x_0^-, x_0) \end{cases}$$

$$\Rightarrow \quad A = \left(\int_{-\infty}^{\infty} u_0^2\right) \left[\frac{G_x(x_0^+, x_0) + G_x(x_0^-, x_0)}{2}\right]$$

Finally, $$\int U_0^3 V_{1y} = \int_{-\infty}^{\infty} \underbrace{U_0^3}_{\text{even}} \Big(\underbrace{\int_0^y u_0^2}_{\text{odd}} + A \Big)\, dy$$

$$= A \int U_0^3$$

$$= \frac{\left(G_x^+ + G_x^-\right)}{2} \left(\int U_0^2\right)\left(\int U_0^3\right)$$

$$\Rightarrow \quad -X_0' = \frac{1}{3V_0^2} \, \frac{\int U_0^2 \int U_0^3}{\int U_{0y}^2} \, \frac{G_x^+ + G_x^-}{2} \, ; \quad \boxed{U_0(y) = V_0 \, \omega(y)}$$

$$= \frac{1}{3} V_0 \, \frac{\int \omega^2 \int \omega^3}{\int \omega_y^2} \, \frac{G_{0x}^+ + G_x^-}{2}$$

$$V_0 = \int u_0^2 \, G_0 \qquad = \left(\int \omega^2\right) V_0^2 G_0$$

$$\Rightarrow \quad V_0 = \frac{1}{\int \omega^2 \, G_0}$$

$$\Rightarrow \quad X_0' = -\left(\frac{G_x^+ + G_x^-}{2 G_0}\right) \frac{\int \omega^3}{3 \int \omega_y^2}$$

$$\frac{G_x^+ + G_x^-}{2G_0} = \frac{\partial_{x_0}\left(\cosh(x_0+L)\cosh(x_0-L)\right)}{2\cosh(x_0+L)\cosh(x_0-L)}$$

$$\cosh(x_0+L)\cosh(x_0-L) = \frac{1}{2}\left(\cosh(2x_0)+\cosh(2L)\right)$$

$$= \frac{\sinh(2x_0)}{\cosh(2x_0)+\cosh(2L)}$$

and $\quad \dfrac{\int w^3}{3\int w_y^2} = 2$

$$\Rightarrow \boxed{\frac{dx_0}{dt} = -\varepsilon^2\, 2\, \frac{\sinh(2x_0)}{\cosh(2x_0)+\cosh(2L)}} \qquad (*)$$

- s.s. at x_0

$$\lambda = \frac{\partial}{\partial x_0}(RHS)\Big|_{x_0=0} = \frac{-4\varepsilon^2}{1+\cosh(2L)} < 0$$

$\Rightarrow x_0 = 0$ is a stable equilibrium!

$\Rightarrow \quad x_0(t) \longrightarrow 0$ as $t\to\infty$

This shows that the interior spike is stable with respect to ~~the~~ slow translations!

- See "gm-dynamics.pde" [flexPDE script], comparing (*) with full numerics.