Homework questions on coarsening

1. Consider an ODE

$$\begin{cases} U_{yy} = U - \dfrac{U^2}{1+\beta U^2}, & y \in \mathbb{R} \\ U_y \to 0 \text{ as } y \to \pm\infty. \end{cases}$$

a) Show that there exists a unique $\beta^\star$ such that U is a heteroclinic orbit when $\beta = \beta^\star$, i.e. U' is monotone for all $y \in \mathbb{R}$.

b) With $\beta = \beta^\star$, let U be a decreasing heteroclinic orbit. Let

$$U^\star = \lim_{y \to -\infty} U(y).$$

Compute $\beta^\star$ and $U^\star$ numerically to at least three significant digits. Hints: $\beta^\star \approx 0.21$ and $U^\star \approx 3.3$ to two significant digits. You may find Maple's fsolve command useful.

2. Consider the Gierer-Meinhardt model with saturation,

$$\begin{cases} u_t = \varepsilon^2 u_{xx} - u + \dfrac{u^2}{v(1+bu^2)}, \qquad \tau v_t = D v_{xx} - v + u^2, \\ 0 = v'(0) = u'(0) = v'(L) = u'(L). \end{cases} \tag{1}$$

Suppose that

$$\varepsilon \ll 1, \;\; D \gg 1.$$

a) Construct a single-interface (half-mesa) solution for the equilibrium problem $u(x,t) = u(x)$. Express the location of the interface l in terms of b and the constants you found in Question 1.b.

b) Find the coarsening thresholds in the regime D is exponentially large.

c) BONUS: Use a computer to simulate the full system (1). Compare your numerical solution with analytical results.