Homework questions on stability of spikes

1. Consider a single spike on interval $[-L, L]$ of the GM system we studied in class,

$$u_t = \varepsilon^2 u_{xx} - u + u^2/v; \quad 0 = v_{xx} - v + u^2, \quad x \in [-L, L] \tag{1}$$

but now impose Dirichlet boundary conditions, $u(\pm L) = 0 = v(\pm L)$. Compute the steady state consisting of a single spike, and study its stability with respect to large and small eigenvalues.

2. (a) Consider the steady state of the GM system (1) with Neumann boundary conditions,

$$0 = \varepsilon^2 u_{xx} - u + u^2/v; \quad 0 = v_{xx} - v + u^2; \quad u'(\pm L) = 0 = v'(\pm L)$$

For a solution that consists of a single spike centered at the origin, compute $v(L)$ and sketch the function $L \to v(L)$. Note that this function has a unique maximum at $L = L_c$. Compute its value.

(b) Suppose that $L_1, \ L_2$ satisfy $v(L_1) = v(L_2)$ with $L_1 < L_c; \quad L_2 > L_c$. Sketch a steady state on the domain of size $L_1 + L_2$ consisting of two asymetric boundary spikes at 0 and at $L_1 + L_2$.

(c) In part (a), you should get that $L_c = arcsinh(1) = 0.8813$. Recall from class that this is precisely the critical threshold at which K small eigenvalues become zero! This is not a coincidence. What's up?

3. In the original paper on stability of K spikes, Iron Ward and Wei set $KL = 1$ but introduced the diffusion conefficient D in front of v_{xx} :

$$u_t = \varepsilon^2 u_{xx} - u + u^2/v; \quad 0 = Dv_{xx} - v + u^2, \quad v'(\pm 1) = 0 = u'(\pm 1).$$

By an appropriate rescaling, this system is equivalent to (1). Find a sequence $D_2 > D_3 > D_4 ...$ such that K spikes on $[-1, 1]$ are stable if and only if $D < D_K$ (with $K \geq 2$).

4. (a) Consider the problem

$$u_t = u_{xx} - u + u^2, \quad x \in [-L, L]; \quad L \gg 1; \quad u'(\pm L) = 0 \tag{2}$$

It has a bump solution centered at the origin whose steady state is approximately given by $u(x) \sim w(x) := \frac{3}{2} sech^2(x/2)$. The corresponding stability problem is

$$\lambda \phi = \phi'' - \phi + 2\phi w$$

It admits a "large" O(1) positive eigenvalue $\lambda_0 > 0$. The next eigenvalue λ_1 is near zero; at the leading order its eigenfunction is $\phi_1 \sim w_x$ and $\lambda_1 \sim Ae^{-2L}$. What is the value of the constant A?

(b) Next consider the problem (2), but with u^2 replaced by $u^2 / \int u^p$ with $p > 1$. This stabilizes the large eigenvalue λ_0. However show that this does not change the small eigenvalue λ_1 that you found in part (a).

(c) Use Maple boundary value problem solver to find λ_1 numerically (this is similar to the worksheet that I gave out to you earlier, ask me if you would like extra help). How does it compare with asymptotics?