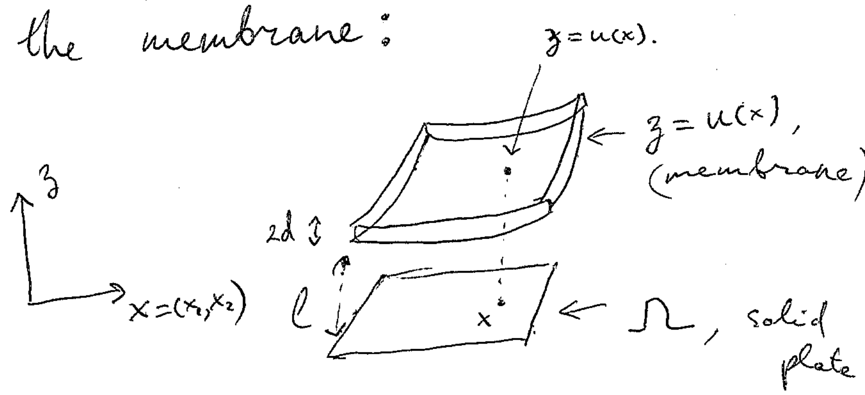
MEMS device modelling

- MEMS [Microelectromechanical system] device consists of an elastic membrane suspended above a rigid plate. A voltage is applied to the top of membrane; the potential difference causes the deflection of the membrane:



Assume:

$d \ll l \ll 1$;

Where
- $2d \equiv$ thickness of membrane
- $l \equiv$ dist. between plate and membrane
- $\Omega \equiv$ geometry of the rigid plate, with $z=0$;
- Membrane $\equiv \{ (x, z) : u(x) - d \leq z \leq u(x) + d \}$

Let $\psi_i \equiv$ electrostatic potential inside membrane;

$\psi_b \equiv$ " " between membrane & plate;

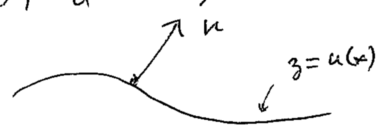
Model:

(1) $\begin{cases} \nabla \cdot (\nabla \psi_b) = 0 \\ \psi_b = 0 \text{ at } z = 0 \end{cases}$; $\begin{cases} \nabla(\varepsilon_2 \nabla \psi_i) = 0 \\ \psi_i = V \text{ at } z = u(x) + d \end{cases}$

(2) $\begin{cases} \psi_b = \psi_i \text{ at } z = u(x) - d \\ \varepsilon_0 \nabla \psi_b \cdot \hat{n} = \varepsilon_2 \nabla \psi_i \cdot \hat{n} \text{ at } z = u(x) - d \end{cases}$

where $\hat{n} \equiv$ normal to $z = u(x) - d$

$\equiv c(-\nabla u, 1)$



$\varepsilon_0 \equiv$ permittivity of free space ; $\varepsilon_0 \equiv$ const.

$\varepsilon_2 \equiv$ " " membrane ; $\varepsilon_2(x, z) = \varepsilon_2(x)$;

Rescale: $\psi_{\substack{i\\b}} = \bar{\psi}_{\substack{i\\b}} V$; $z = \bar{z}\, l$

Drop the bars:

$$\Delta_x \psi_b + \frac{1}{l^2}\psi_{b_{zz}} = 0 , \quad \nabla_x(\varepsilon_2 \nabla_x \psi_i) + \frac{\varepsilon_2}{l^2}\psi_{i_{zz}} = 0$$

$\psi_b = 0$ at $z = 0$, $\psi_i = 1$ at $z = u(x) + \frac{d}{l}$

$\psi_b = \psi_i$ at $z = u(x) - \frac{d}{l}$

$$-\nabla_x u \cdot \nabla_x\left(\psi_b - \frac{\varepsilon_2}{\varepsilon_0}\psi_i\right) + \frac{1}{l}\left(\psi_{b_z} - \frac{\varepsilon_2}{\varepsilon_0}\psi_{i_z}\right) = 0$$

Let $\delta = \frac{d}{l} \ll 1$; then at leading order we get:

$$(3) \quad \begin{cases} \psi_{b_{zz}} = 0 = \psi_{i_{zz}} \\ \psi_b = 0 \text{ at } z = 0, \quad \psi_i = 1 \text{ at } z = u + \delta \\ \frac{\varepsilon_2}{\varepsilon_0}\psi_{i_z} = \psi_{b_z} \text{ at } z = u - \delta \end{cases}$$

$$(4) \Rightarrow \begin{cases} \psi_b = A z \\ \psi_i = 1 + B(z - (u+\delta)) \end{cases}$$

and $A(u - \delta) = 1 + B(\underbrace{u - \delta - (u+\delta)}_{2\delta})$

$A = \frac{\varepsilon_2}{\varepsilon_0} B$

$$\Rightarrow \quad \psi_i \sim 1 + \frac{z-u}{\frac{\varepsilon_2}{\varepsilon_0} u} \quad ; \quad \psi_b \sim z/u$$

Deflection equation:

$$(5) \qquad T \Delta u = \frac{\varepsilon_2}{2} \left| \nabla \psi_i \right|^2$$

$$\nabla \psi_i = \left|\nabla_x \psi_i\right|^2 + \underbrace{\left|\frac{V}{\ell} \psi_{i_z}\right|^2}_{\frac{1}{\ell^2} \frac{\varepsilon_0^2}{\varepsilon_2^2} \frac{1}{u^2}} \qquad \text{[after rescaling]}$$

$\Rightarrow$

$$(6) \qquad \boxed{\begin{cases} \Delta u = \dfrac{\lambda f(x)}{u^2}, & x \in \Omega \\ u = 1, & x \in \partial\Omega \end{cases}}$$

where $f(x) = \frac{\varepsilon_0^2}{\varepsilon_2^0}$; $\lambda = \frac{\varepsilon_0 V^2}{2\ell^2 T}$

Note that $0 \leq f(x) \leq 1$.

- Experimentally, it is observed that the MEMS device becomes unstable if voltage is too high, and the membrane collapses. This is known as a "pull-in" voltage instability.
- Mathematically, solution to (6) ceases to exist if $\lambda > \lambda^*$ for some λ^*, the "pull-in" value.

Theorem 1: Suppose that $0 < c \leq f(x) \leq 1$. Then there exists λ^* such that (6) has no solution if $\lambda > \lambda^*$. Moreover,

$$\lambda^* \geq \frac{4\alpha_1}{27c} \quad \text{where } \alpha_1 \text{ is}$$

the lowest eigenvalue of

$$(7) \quad \begin{cases} -\Delta\varphi = \alpha\varphi & \text{inside } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof: Rewrite (6) as: by shifting $u \to u-1$:

$$(8) \quad \begin{cases} \Delta u = \dfrac{\lambda f(x)}{(1+u)^2}, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

Let (φ_1, α_1) be principal eigenfunction / eigenvalue pair to (7). Then we can assume $\varphi_1 > 0$ inside Ω and $\alpha_1 > 0$. Multiply (8) by φ_1 & integrate by parts, we get:

$$(9) \quad \int_\Omega \varphi\left(\alpha_1 u + \frac{\lambda f(x)}{(1+u)^2}\right) = 0$$

Since $\varphi > 0$, the expression in brackets must change sign in order to satisfy (9).

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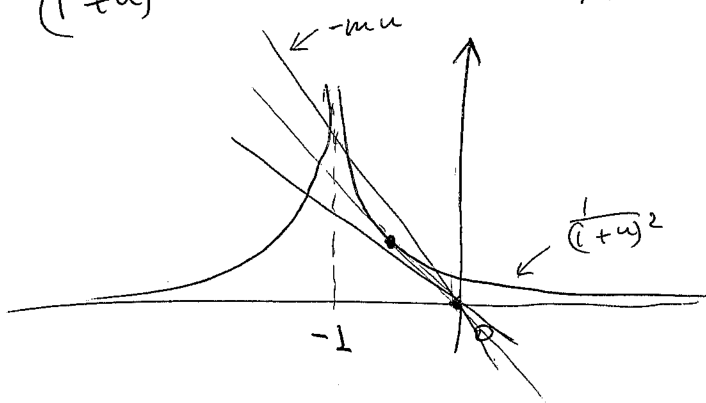
Now $\quad (\,) \geq \alpha_1 u + \frac{\lambda c}{(1+u)^2}$

So consider $\quad m u + \frac{1}{(1+u)^2} = 0, \quad m = \frac{\alpha_1}{\lambda c}. \; (10)$



or $\quad -mu = \frac{1}{(1+u)^2}$. The double tangency occurs if $-m = \frac{-2}{(1+u)^3} \Rightarrow -\frac{(1+u)}{2} = u$

$$\Rightarrow u = -\frac{\frac{1}{2}}{1+\frac{1}{2}} = -\frac{1}{3}$$

$$\Rightarrow m = \frac{2}{\left(\frac{2}{3}\right)^3} = \frac{27}{4}$$

<u>Thus</u> (10) has <u>no sol'n with $u > -1$</u> if $m < \frac{27}{4}$

or $\boxed{\lambda > \frac{27c}{4\alpha_1}}$

<u>Conclusion</u>: the pull-in instability exists for any choice of the dielectric profile $f(x)$.

Next we show that sol'n to (6) exists if λ is sufficiently small, using sub-supersolutions.

Def: A fcn $\bar{u}$ is <u>upper sol'n</u> if

$$(11) \quad \begin{cases} \Delta \bar{u} \le \dfrac{\lambda f(x)}{(1+\bar{u})^2} & \text{inside } \Omega \\ \bar{u} \ge 0 & \text{on } \partial\Omega . \end{cases}$$

It is <u>lower solution</u> if inequalities in (11) are reversed.

So any positive constant is an upper solution.

To find a <u>lower solution</u>, consider the principal eigfcn/value of:

$$(12) \quad \begin{cases} -\Delta \varphi = \mu \varphi & \text{inside } \Omega' \\ \varphi = 0 & \text{on } \partial\Omega' \end{cases}$$

Where $\Omega' \supset \Omega$ to be specified later; also take $\varphi > 0$ inside Ω' with $\max_{\Omega'} \varphi = 1$.

Our lower sol'n candidate is:

$$\underline{u} = -a\varphi \quad , \quad a > 0 ,$$

so that $\underline{u} \leq 0$ on $\partial\Omega$ is satisfied.

Now $\Delta \underline{u} = \mu a \varphi$ so that

$$\Delta \underline{u} \geq \frac{\lambda f}{(1+\underline{u})^2} \quad \text{becomes}$$

$$(*) \qquad \frac{\mu}{\lambda} a\varphi(x) \geq \frac{1}{(1-a\varphi(x))^2} , \qquad [\text{ we assume } 0 < f < 1]$$

this must hold for all $x \in \Omega$.

Moreover, we must have $1 + \underline{u} \geq 0$

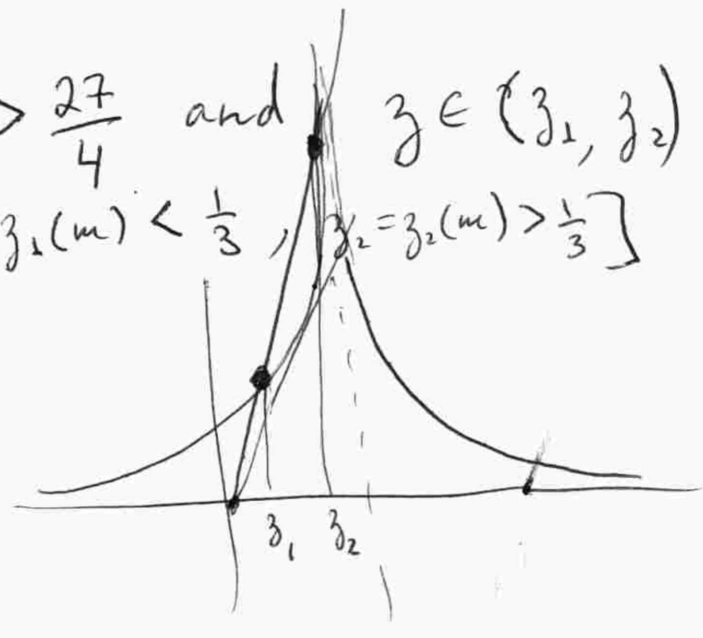
so we write $(*)$ as:

$$(**) \qquad m z \geq \frac{1}{(1-z)^2} \quad ; \quad m = \frac{\mu}{\lambda} ,$$

$$z = a\varphi(x)$$

$$0 < z < 1 .$$

Now $(**)$ holds if $m > \frac{27}{4}$ and $z \in (z_1, z_2)$
[where $z_1 = z_1(m) < \frac{1}{3}$, $z_2 = z_2(m) > \frac{1}{3}$]



~~We can~~

- Let's choose $\boxed{z_1 = \frac{1}{4}}$ [any choice $< \frac{1}{3}$ is ok]
- Then $m = \frac{1}{z_1} \frac{1}{(1-z_1)^2} = \frac{64}{9} = 7.111$
- Then z_2 satisfies: $z_2(1-z_2)^2 = z_1(1-z_1)^2$ [with $z_1 < z_2 < 1$]

$$\to z_2 = 1 - \frac{z_1}{2} - \frac{\sqrt{4z_1 - 3z_1^2}}{2}$$

$$\boxed{z_1 = 0.25, \ \boxed{z_2 = 0.4736}, \ m \geq 7.111}$$

Then $\underline{u}$ is a lower sol'n if

$$z_1 \leq a\,\varphi(x) \leq z_2 \quad \forall\, x \in \Omega$$

Choose $a = \frac{z_2}{\max_\Omega \varphi}$. Then $\underline{u}$ is subsol'n provided that

$$z_1 \leq z_2 \frac{\varphi(x)}{\max_\Omega \varphi}$$

$$\iff \quad \frac{\min_\Omega \varphi}{\max_\Omega \varphi} \geq \frac{z_1}{z_2} = 0.589197$$

and in addition, $m = \frac{\mu}{\lambda} \geq 7.111$

<u>Conclusion</u>: <u>Thm 2</u>: Suppose $0 \le f(x) \le 1$,

and suppose that
$$\lambda \le \frac{\mu}{7.111} = \frac{9}{64}\mu,$$

where μ is ~~an~~ principal eigenvalue of
$$\begin{cases} \Delta\varphi = -\mu\varphi & \text{on } \Omega' \supset \Omega \\ \varphi = 0 & \text{on } \Omega' \end{cases}$$

and in addition,
$$\frac{\min_\Omega \varphi}{\max_\Omega \varphi} \geqslant 0.5892.$$

Then sol'n to (6) exists.

<u>HW</u>: Show that $\exists\,\mu$ (and Ω') which satisfies the conditions of thm 2.

Scaling analysis

Let's consider the radially symmetric case

$n=1$: $\Omega = (-L, L)$

$n=2$: $\Omega = B_1(0) = \{x : |x|^2 < L^2\}$

and $f(x) = 1$. Then (6) becomes:

$$(16) \quad \begin{cases} u_{rr} + \frac{n-1}{r} u_r = \frac{\lambda}{u^2}, & 0 < r < L \\ u'(0) = 0, \quad u(L) = 1. \end{cases}$$

We rescale: $\begin{cases} u(r) = a\, w(\eta) \\ \eta = b r \end{cases}$

$$\Rightarrow \begin{cases} w_{\eta\eta} + \frac{n-1}{\eta} w_\eta = \frac{\lambda}{b^2 a^3} w^{-2} \\ w'(0) = 0, \quad w(bL) = 1/a \end{cases}$$

Now choose $\boxed{a = u(0)}$; $\frac{\lambda}{b^2 a^3} = 1$.

Then (16) becomes:

$$(17) \quad \begin{cases} (a) & w_{\eta\eta} + \frac{n-1}{\eta} w_\eta = w^{-2} \\ (b) & w'(0) = 0, \quad w(0) = 1 \end{cases}$$

along with:

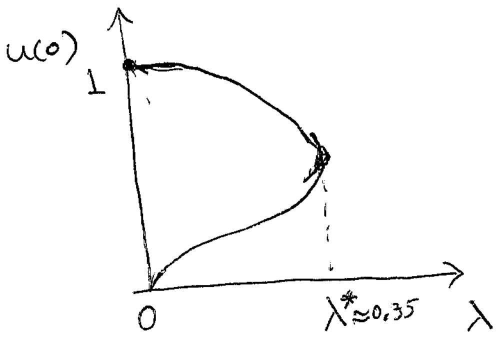
$$(18) \quad \begin{cases} a = u(0) = 1/w(bL) \\ \lambda = \frac{b^2}{[w(bL)]^3} \end{cases} ; \quad u(r) = w(br)$$

In other words, we have converted a boundary value problem (16) into an initial value problem (17)!

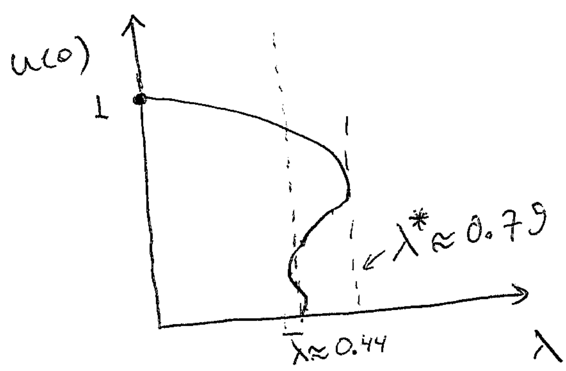
Numerically, (17) is much easier to solve than (16). This method is possible because of the underlying scaling symmetry of (16).

Using (17, 18), we can now draw a <u>bifurcation diagram</u> of (16), plotting $a = u(0)$ v.s. λ [parametrized by "b"]. Using Maple we get; with $L = 1$:

$n = 1$:



$n = 2$



This method gives λ^* numerically: $n=1: \lambda^* \approx 0.35$; $n=2: \lambda^* \approx 0.79$

- When $n = 1$, we observe that $\lambda \to 0$, $u(0) \to 0$ as $b \to \infty$
- When $n = 2$, we see that $\lambda \to \bar{\lambda} \approx 0.44$ as $b \to \infty$

  Moreover, the bifurcation curve seems to oscillate as $b \to \infty$, $\lambda \to \bar{\lambda}$. So <u>there exists infinitely many solutions</u> to (16) when $n = 2$, $\lambda = \bar{\lambda}$.

Can we show this?

• Note that (17a) has a scaling invariance
$\eta = \alpha \hat{\eta}$, $w = \beta \hat{w}$, whenever $\beta = \alpha^{2/3}$.

This means that a reduction of order is possible via a change of variables:

$$(19) \qquad \xi = \ln \eta, \quad w = \eta^p V$$

where p is to be specified. We compute:

$$\eta = e^{\xi}; \quad w = e^{p\xi} V$$
$$w_\eta = e^{-\xi + p\xi}(pV + V_\xi)$$
$$w_{\eta\eta} = e^{(-2+p)\xi}\left(p(p-1)V + (2p-1)V_\xi + V_{\xi\xi}\right)$$

$$\Rightarrow e^{(-2+p)\xi}\left\{V_{\xi\xi} + [(2p-1) + (n-1)]V_\xi + [p(p-1) + (n-1)p]V\right\}$$
$$= e^{-2p\xi} V^{-2}$$

$\Rightarrow$ choose $-2+p = -2p \Rightarrow \boxed{p = \frac{2}{3}}$

Then the indep. variable dissapears, and we get a 2-nd order dynamical system. To simplify further, let $p = \frac{1}{V}$, $q = \frac{V_\xi}{V}$; we get:

$$(20) \qquad \begin{cases} \dfrac{dp}{d\xi} = -pq \\ \dfrac{dq}{d\xi} = -q^2 + p^3 - [n - \frac{2}{3}]q - [\frac{2}{3}(n - \frac{4}{3})] \end{cases}$$

In terms of η, w we have:

$$(21) \qquad \eta = e^{\xi}; \quad p = \frac{\eta^{2/3}}{w}; \quad q = \eta \frac{w_\eta}{w} - \frac{2}{3}$$

So initial conditions (17 b) become:

$$(22) \qquad \xi \to -\infty; \quad p \to 0, \quad q \to -\frac{2}{3}.$$

Now consider the phase portrait of (20):

- Steady states are: A: $p = 0$, $q = -\frac{2}{3}$

  B: $p = 0$, $q = \frac{4-3n}{3}$

  C: $p = \left[\frac{2}{3}\left(n - \frac{4}{3}\right)\right]^{\frac{1}{3}}$, $q = 0$

Note that A corresponds precisely to (22)!

Linearize (20):

$$J = \begin{pmatrix} -q & -p \\ 3p^2 & -2q - \left(n - \frac{2}{3}\right) \end{pmatrix}$$

$$J|_A = \begin{pmatrix} 2/3 & 0 \\ 0 & 2-n \end{pmatrix} \qquad J|_B = \begin{pmatrix} \frac{-4+3n}{3} & 0 \\ 0 & \frac{-3n+2}{3} \end{pmatrix}$$

$$J|_C = \begin{pmatrix} 0 & -P_0 \\ 3P_0^2 & -n + \frac{2}{3} \end{pmatrix}, \quad \text{where } P_0 = \left(\frac{2}{3}\left(-\frac{4}{3} + n\right)\right)^{\frac{1}{3}}.$$

Treat n as a bifurcation parameter; $n \geq 1$.

Stability: A: $\lambda = \frac{2}{3}$, $2-n$

- Source if $n < 2$
- Saddle if $n > 2$

B:

- Sync if $n < \frac{4}{3}$
- Saddle if $n > \frac{4}{3}$

C: $\det J_c = 3p_0^3 = 2(-\frac{4}{3}+n)$ $\begin{cases} <0, & n<\frac{4}{3} \\ >0, & n>\frac{4}{3} \end{cases}$

$\mathrm{tr}\, J_c = \frac{2}{3} - n < 0$

$\mathrm{tr}^2 - 4\det = n^2 - \frac{28}{3}n + \frac{100}{9}$ $\begin{cases} >0, & n=1 \\ <0, & n=2,\dots,7 \\ >0, & n \geq 8 \end{cases}$

So C is
- Saddle if $n=1$
- Spiral sync if $n = 2, 3 \dots 7$
- Sync if $n \geq 8$

See phase plots for $n=1, 2$ on next page.

Now we want to know what happens to (18) as $b \to \infty$, i.e. behaviour of $\omega(\eta)$ as $\eta \to \infty \Rightarrow \xi \to \infty$.

Moreover, $\omega > 0$ so $p > 0$. From phase plane, any sol'n that starts with A as $\xi \to -\infty$, must end up at B as $\xi \to +\infty$ (if $n=1$) or at C as $\xi \to +\infty$ (if $n \geq 2$)

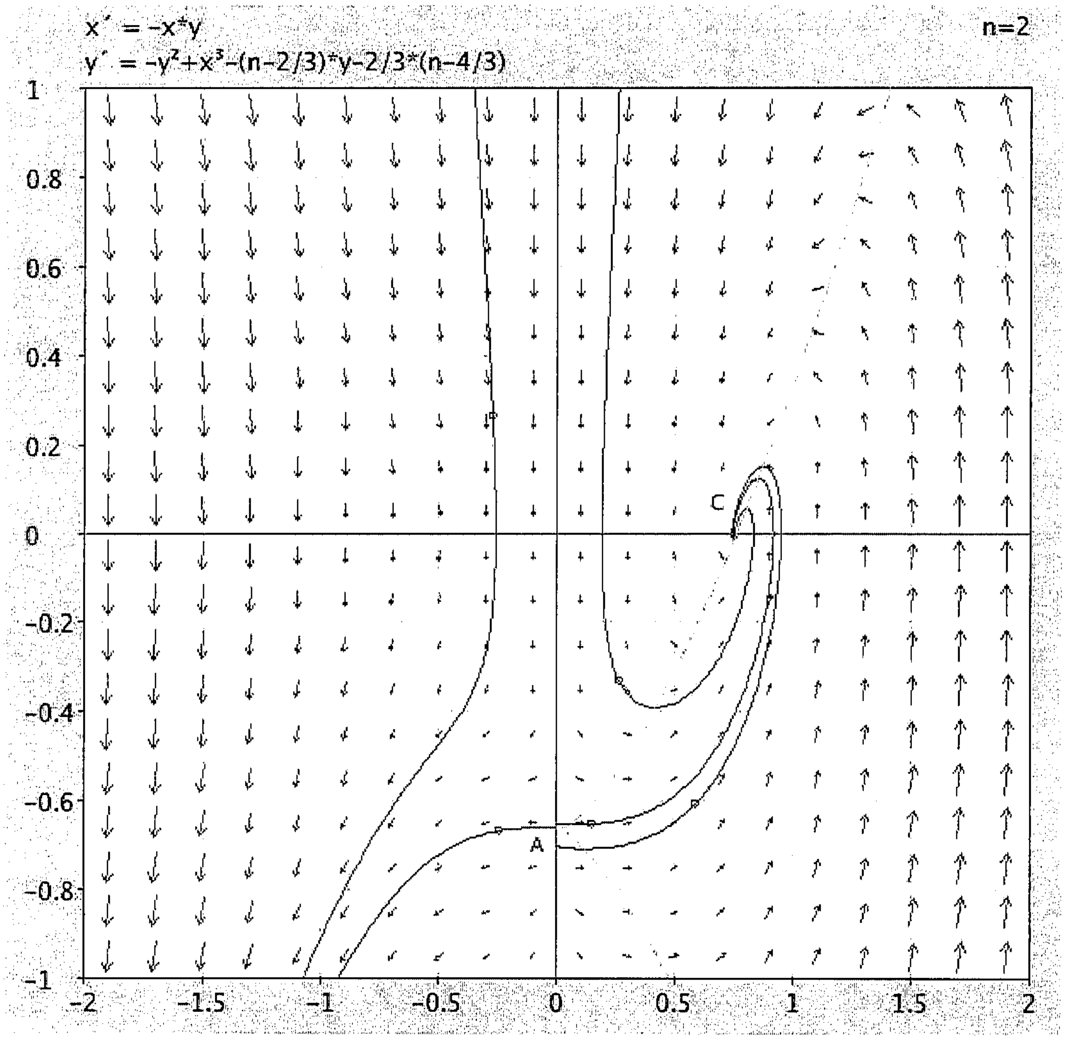
If $n=1$, near B we get

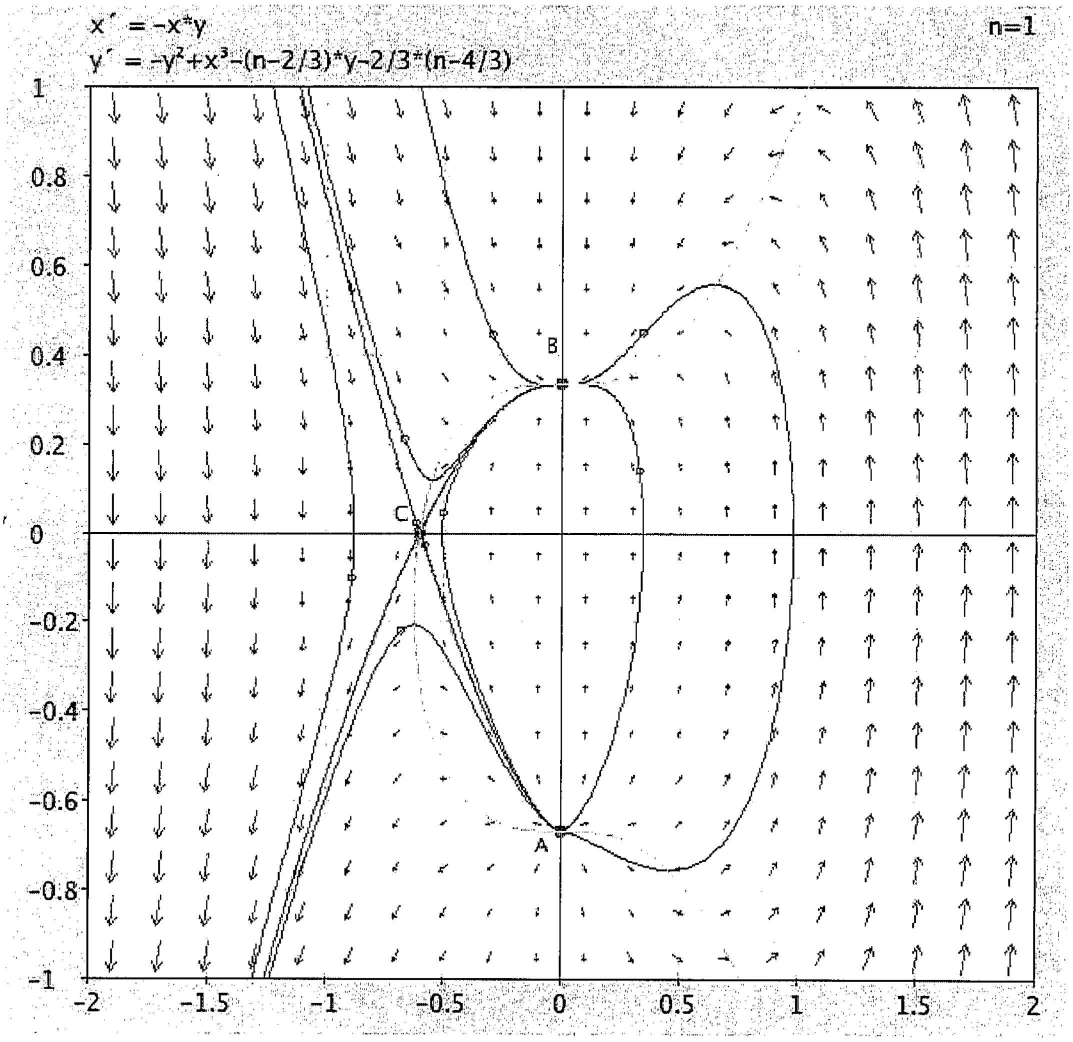
$$\begin{pmatrix} p \\ q \end{pmatrix} \sim C_1 e^{-\frac{1}{3}\xi} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-\frac{4}{3}\xi} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \beta\omega \sim C_0 \eta$ as $\eta \to \infty$

$\Rightarrow$ $\left\{ a \sim \frac{1}{C_0 bL}, \quad \lambda \sim \frac{1}{C_0^3 b L^2} \right.$ as $b \to 0$

So $\lambda \to 0$, $u(0) \to 0$ (see figure with $n=1$ on p. 9)


x´ = −x*y
y´ = −y²+x³−(n−2/3)*y−2/3*(n−4/3)
n=2
A
C
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2


x´ = −x*y
y´ = −y²+x³−(n−2/3)*y−2/3*(n−4/3)
n=1
A
B
C
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2

If $n=2$, near C we get:

$$\lambda = -\frac{2}{3} \pm i\, 1.4337$$

$$\Rightarrow \quad p \sim e^{-\frac{2}{3}\xi}\left[A\cos(1.43\,\xi) + B\sin(1.43\,\xi)\right] + \left(\frac{4}{9}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad w(\eta) \sim \eta^{\frac{2}{3}}\left(\frac{9}{4}\right)^{\frac{1}{3}} + \text{some small oscillations}$$

$$\Rightarrow \begin{cases} \lambda \sim \frac{1}{L^2}\frac{4}{9} + \text{some oscillations as } b \to \infty \\ u(0) \sim c\left(b\right)^{-\frac{2}{3}} \to 0 \quad \text{as } b \to 0 \end{cases}$$

This shows that $\bar{\lambda} = \frac{4}{9} \simeq 0.44$ in figure $n=2$) of page 9.

Some questions

1) Consider condition (15) of theorem 2. Can (15) be satisfied?

a) If $\bar{\Omega} = [-\ell, \ell]$, show that

$$\mu = \frac{\alpha}{\theta_0^2} \quad \text{where} \quad \cos\theta_0 = \frac{1}{3}$$

and α as in theorem 1. Conclude that

if $f(x) = 1$ then $\lambda^* \in \frac{4}{27}\frac{\pi^2}{\ell^2}\left[\frac{1}{\theta_0^2}, 1\right]$.

b) If $\Omega \subset \mathbb{R}^2$, how can you choose Ω' to satisfy (15)?

2) Consider the problem:

$$(*) \quad \begin{cases} u_{rr} + \frac{n-1}{r} u_r + \lambda e^{u} = 0 \\ u'(0) = 0, \quad u(1) = 0, \quad u > 0 \text{ in } [0,1). \end{cases}$$

Sketch the bifurcation diagram $u(0)$ vs. λ for $n = 1, 2, 3$. Show that $(*)$ has infinitely many solutions if $n = 3$ and $\lambda = 2$, but at most two solutions for any λ if $n \leq 2$.

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