Multiple scales:

- Using complex variables

Ex: $u_{tt} + u = \varepsilon u^2$

$u_{tt} = U_{tt} + 2U_{ts}\varepsilon + U_{ss}\varepsilon^2$

Expand: $u = U(t, s)$ $\quad s = \varepsilon t$

$U = U_0 + \varepsilon U_1 + \cdots$

$U_{0\,tt} + U_0 = 0$

$U_{1\,tt} + U_1 = U_0^2 - 2U_{0\,ts}$

$U_0 = A e^{it} + \bar{A} e^{-it}$ [since u_0 real]

A: complex, $A = A(s) \in \mathbb{C}$

$= A e^{it} + \underbrace{c.c.}_{\text{"complex conjugate"}}$

$U_0^2 = A^2 e^{2it} + 2A\bar{A} + \bar{A}^2 e^{-2it}$

$U_{0\,ts} = iA_s e^{it} + c.c.$

$u_{1\,tt} + u_1 = e^{it}\left(\boxed{-2iA_s}\right) + e^{-it}(2i\bar{A}_s) + \ldots$

$+ e^{2it}(\ ) + e^{0it} \cdots$

Eliminate resonance: $A_s = 0 \Rightarrow A(s) = A$

- No εt scale
- Need $\varepsilon^2 t$ scale to determine amplitude + frequency corrections.

Let $\tau = \varepsilon^2 t$

$$U = U(t, \overset{\varepsilon t}{s}, \overset{\varepsilon^2 t}{\tau})$$

$$\frac{d^2}{dt^2} U = U_{tt} + 2\varepsilon U_{ts} + 2\varepsilon^2 U_{t\tau} + \varepsilon^2 U_{ss} + 2\varepsilon^3 U_{s\tau} + \varepsilon^4 U_{\tau\tau}$$

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2$$

$u_{tt} + u = \varepsilon u^2$

$$U_{0\,tt} + U_0 = 0$$

$$u_0 = A e^{it} + c.c.$$

$$\boxed{A_s = 0} \Rightarrow A = A(\tau)$$

$$U_{1\,tt} + U_1 = U_0^2 - \cancel{2U_{0ts}}$$

$$= A^2 e^{2it} + 2A\bar{A} + \bar{A}^2 e^{-2it}$$

Solve for u_1:

$$U_1 = \frac{A^2}{1-4} e^{2it} + 2A\bar{A} + \frac{\bar{A}^2}{1-4} e^{-2it} + (B e^{it} + c.c.)$$

$[\, B_s = 0 \,]$

$O(\varepsilon^2)$:

$$u_{2tt} + u_2 = 2u_0u_1 - \cancel{2u_{1ts}} - 2u_{0t\tau} \quad \cancel{-u_{0ss}}$$

0

0

Why $B_s = 0$?

- I.C. $u(0) = a = u_0 + \varepsilon u_1 + \dots$
  $u'(0) = 0$

$$\Rightarrow \begin{array}{l} u_0 = a \text{ at } t=0 \\ \left.\begin{array}{l} u_1 = 0 \text{ at } t=0 \\ u_1' = 0 \text{ at } t=0 \end{array}\right\} \Rightarrow B = \text{fcn of } A(\tau) \\ \qquad\qquad\qquad\qquad \Rightarrow B \equiv B(\tau) \end{array}$$

$$u_{2tt} + u_2 = 2u_0u_1 - 2u_{0t\tau}$$

$iA_\tau e^{it}$ + c.c.

$$u_0 = Ae^{it} + \text{c.c.}$$

$$u_1 = 2A\bar{A} - \frac{1}{3}A^2e^{2it} - \frac{1}{3}\bar{A}^2e^{-2it}$$

$$\Rightarrow u_{2tt} + u_2 = e^{it}\left\{2\left(-\frac{1}{3}A^2\bar{A} + 2A^2\bar{A}\right) - 2(A_\tau i)\right\} + e^{-it}\{\text{c.c.}\} + \dots$$

Eliminate secular (resonant) terms:

$$\boxed{\underbrace{\left(-\frac{1}{3} + 2\right)}_{5/3}A^2\bar{A} - iA_\tau = 0}$$

$u \sim A e^{it} + \bar{A} e^{-it}$

Let $A = R e^{i\varphi}$, $R(\tau), \varphi(\tau) \in \mathbb{R}$

then $u = R e^{i(t+\varphi)} + R e^{-i(t+\varphi)}$

$= 2R\cos(t+\varphi)$

$A_\tau = \left(R_\tau + iR\varphi_\tau\right) e^{i\varphi}$

$\left(R_\tau + iR\varphi_\tau\right) i = +\frac{5}{3} R^3$

$\Rightarrow -R\varphi_\tau = \frac{5}{3} R^3$, $R_\tau = 0$

$\boxed{\varphi_\tau = -\frac{3}{5} R^2}$

$\Rightarrow \boxed{u = 2R\cos\left(t\left(1 - \frac{3}{5}R^2\varepsilon^2\right)\right)}$