Theorem: Let $w(x) := \frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)$ be the ground-state solution to

$$\begin{cases} w'' - w + w^2 = 0, \quad w'(0) = 0, \quad w \to 0 \text{ as } |x| \to \infty \\ w > 0. \end{cases}$$

Let $L_0 \varphi := \varphi'' - \varphi + 2\varphi w.$

Consider the non-local eigenvalue problem:

$$(\text{NLEP}) \quad \begin{cases} L_0 \varphi - \dfrac{\gamma \left(\int w \varphi\right)}{\int w^2} w^2 = \lambda \varphi \\ \\ \varphi \to 0 \quad \text{as} \quad |x| \to \infty \end{cases}$$

(a) Suppose that $\gamma < 1$. Then $\exists\, \lambda$ with $\lambda > 0$

(b) Suppose that $\gamma > 1$. Then either $\operatorname{Re}(\lambda) < 0$.

or else λ=0, ϕ=w'.

Lemma 1 [Facts about L_0]

(a) The Local eigenvalue pblm $L_0 \varphi = \lambda \varphi$ has unique positive eigenvalue $\lambda = \frac{1}{4}$,

(b) $\operatorname{Ker} L_0 = \operatorname{span}\{w'\}$

(c) $L_0 w = w^2$; $L\left(w + \frac{x w'}{2}\right) = w$

(d) $\int w = \int w^2 = 6$, $\int w^3 = \frac{36}{5}$, $\int w'^2 = \frac{6}{5}$

Proof: (b): $L_0 w' = 0$ is obvious, the converse follows from uniqueness of solutions to ODE's:

$L_0 \varphi = 0$ has two independent solutions; one of them is $\varphi = w'$; the other will blow up at $x \to \infty$.

(a) Since L_0 is self-adjoint, by oscillation theorem, its eigenvalues are indexed by the number of roots of the eigenfunctions. Now w' has one root so $\lambda = 0$ is a second eigenvalue, $\Rightarrow \exists! \lambda > 0$. The exact value $\lambda = \frac{1}{4}$ can be computed using hypergeometric functions.

(c, d): Direct computation. $\blacksquare$

Proof of Theorem (part a): We will look for $\lambda > 0$.

Assume that $\int \varphi \omega \neq 0$.

$$\begin{cases} L_0 \varphi - \lambda \varphi = \omega^2 \\ \dfrac{\gamma \int \omega \varphi}{\int \omega^2} = 1 \end{cases}$$

or

$$\underbrace{\int \omega (L_0 - \lambda)^{-1} \omega^2}_{f(\lambda)} = \frac{\int \omega^2}{\gamma}$$

Now $f(0) = \int \omega \underbrace{L^{-1} \omega^2}_{\omega} = \int \omega^2$

so that $f(0) > \dfrac{\int \omega^2}{\gamma}$ when $\gamma < 1$.

On the other hand, $f(\lambda)$ has a vertical asymptote at $\lambda = \lambda_0$ where $\lambda_0 = \frac{1}{4}$ is the eigenvalue of L_0. Moreover, we claim that $f(\lambda) \to -\infty$ as $\lambda \to \lambda_0^-$.

Proof of claim: Let $\lambda = \lambda_0 - \delta$, $\delta \ll 1$.

Then $L_0 \varphi - \lambda_0 \varphi - \delta \varphi = \omega^2$

Project: $\omega^2 = a \varphi_0 + R$ where

φ_0 is the eigenfunction $L \varphi_0 = \lambda_0 \varphi_0$, normalized so that $\int \varphi_0^2 = 1$

and $a = \int \omega^2 \varphi_0$; $R \perp \varphi_0$.

Then $\varphi \sim -\frac{a}{\delta}\varphi_0 + O(1)$

so that $f(\lambda_0-\delta) \sim -\frac{a}{\delta}\int \omega\varphi_0 \sim -\frac{\int\omega^2\varphi_0 \int\omega\varphi_0}{\delta}$

But φ_0 has no roots [since it's the leading eigenfunction of the local operator] $\Rightarrow f(\lambda_0-\delta) \to -\infty$ as $\delta \to 0^+$.

Thus $\exists\ \lambda \in (0, \lambda_0)$ s.t. $f(\lambda) = \frac{\int\omega^2}{\gamma}$ ∎

Before proving part (b), we first ~~to~~ need the following lemma.

Lemma 2 Consider the operator

$$L_1\varphi := L_0\varphi - \frac{\int\omega\varphi}{\int\omega^2}\omega^2 - \frac{\int\omega^2\varphi}{\int\omega^2}\omega + \frac{\int\omega^3}{(\int\omega^2)^2}\left(\int\omega\varphi\right)\omega.$$

Then: (a) $\operatorname{Ker} L_1 = \{\omega, \omega'\}$ and L_1 is self-adjoint.

(b) $\int\varphi L_1\varphi < 0$ for all $\varphi\in C^2(\mathbb{R})$.

Proof: Note that $L_1\omega' = L_0\omega' = 0$

and $L_1\omega = \omega^2 - \omega^2 - \frac{\int\omega^3}{\int\omega^2} + \frac{\int\omega^3}{\int\omega^2} = 0$

so certainly $\omega, \omega' \in \ker L_1$.

Now suppose that $L_1\varphi = 0$ for some φ.

Then $L_0\varphi = c_1\omega + c_2\omega^2$ where

$$c_1 = \frac{\int\omega^2\varphi}{\int\omega^2} - \frac{\int\omega^3}{(\int\omega^2)^2}\int\omega\varphi, \quad c_2 = \frac{\int\omega\varphi}{\int\omega^2}$$

Now $\omega^2 = L_0\omega$; $\omega = L_0\left(\omega + \frac{x\omega'}{2}\right)$

$$\Rightarrow L_0\left(\varphi - c_1\left(\omega + \frac{x\omega'}{2}\right) - c_2\omega\right) = 0$$

$$\Rightarrow \varphi = c_1\left(\omega + \frac{x\omega'}{2}\right) + c_2\omega + c_3\omega'$$

But then $L_1 c_1\left(\frac{x\omega'}{2}\right) = 0$ since $\omega, \omega' \in \ker L_1$.

Next we compute

$$L_1\left(\frac{x\omega'}{2}\right) =$$

$$= \omega - \omega^2 + \frac{\int \omega x \omega'}{2\int \omega^2}\left(-\omega^2 + \frac{\int \omega^3}{\int \omega^2}\omega\right) - \frac{\int \omega^2 x \omega'}{2\int \omega^2}\omega$$

$$\int \omega x \omega' = -\int \frac{\omega^2}{2}, \quad \int \omega^2 x \omega' = -\int \frac{\omega^3}{3}.$$

$$L_1\left(\frac{x\omega'}{2}\right) = \omega\left(1 - \frac{\int \omega^3}{(\int \omega^2)^2}\frac{\int \omega^2}{4}\right) + \omega^2\left(-1 + \frac{\int \omega^3}{6\int \omega^2}\right)$$

$$\neq 0$$

Thus $c_1 = 0 \Rightarrow \varphi \in \text{span}\{\omega, \omega'\}$.

It is clear that $\int \psi L_1 \varphi = \int \varphi L_1 \psi$

so that L_1 is self-adjoint [by construction].

Proof (b): Since L_1 is self-adjoint, its eigenvalues are all real. Thus to show part (b) it suffices to show that L_1 has no strictly positive eigenvalue. We proceed by contradiction. Suppose

$$L_1 \varphi = \lambda \varphi \quad \text{with} \quad \lambda > 0.$$

Since $L_1 \omega = 0$, assume WLOG that $\varphi \perp \omega$.

Then $$(L_0 - \lambda)\varphi = \frac{\int \omega \varphi}{\int \omega^2}\omega^2$$

$$\Rightarrow \left(\int \omega^2\right)(L-\lambda)\,\varphi = \left((L-\lambda)\omega + \lambda\,\omega\right)\int \omega\,\varphi$$

$$\int \omega \left\{ \varphi \int \omega^2 = \left[\omega + \lambda (L-\lambda)^{-1}\omega\right] \int \omega\,\varphi \right\}$$

$$\Rightarrow \quad \int \left((L-\lambda)^{-1}\omega\right)\omega = 0$$

Now ~~do~~ let $f(\lambda) = \int \omega (L_0-\lambda)^{-1}\omega$.

We have: $f(0) = \int \omega\left(\omega + \frac{x\omega'}{2}\right) = \left(\int \omega^2\right)\left(1 - \frac{1}{4}\right) > 0$

and $f'(\lambda) = \int \omega (L-\lambda)^{-2}\omega = \int \left\{(L-\lambda)^{-1}\omega\right\}^2 > 0$

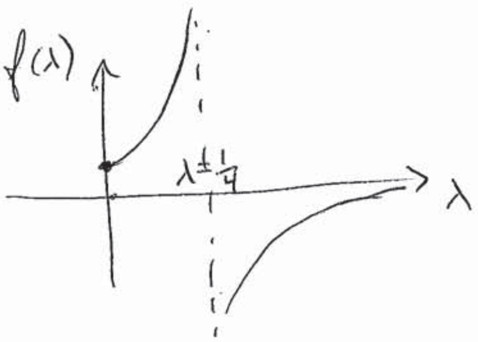
So $f(\lambda)$ is increasing.

Also $f(\lambda)$ has a singularity at $\lambda = \frac{1}{4}$

[the eigenvalue of the local operator L_0]

and moreover $f(\lambda) \to 0$ as $\lambda \to \infty$

[since $f(\lambda) \sim -\int \frac{\omega^2}{\lambda}$ as $\lambda \to \infty$].



This shows that $f(\lambda) \neq 0 \ \forall \lambda > 0$

$\Rightarrow L_1$ has no positive eigenvalues. ∎

<u>Proof of Theorem (part b)</u> In general, λ and φ can be complex. So we write:

$$\lambda = \lambda_R + i\lambda_I, \quad \varphi = \varphi_R + i\varphi_I$$

and

$$\text{(a)} \quad L_0 \varphi_R - \frac{\gamma}{\int \omega^2}\left(\int \varphi_R \omega\right)\omega^2 = \lambda_R \varphi_R - \lambda_I \varphi_I$$

$$\text{(b)} \quad L_0 \varphi_I - \frac{\gamma}{\int \omega^2}\left(\int \varphi_I \omega\right)\omega^2 = \lambda_R \varphi_I + \lambda_I \varphi_R$$

Now take $\int (a)\varphi_R + \int (b)\varphi_I$

$$\int \varphi_R L_0 \varphi_R + \int \varphi_I L_0 \varphi_I - \frac{\gamma}{\int \omega^2}\left[\left(\int \varphi_R \omega\right)\left(\int \varphi_R \omega^2\right) + \left(\int \varphi_I \omega\right)\left(\int \varphi_I \omega^2\right)\right]$$
$$= \lambda_R \left[\int \varphi_R^2 + \int \varphi_I^2\right]$$

Also note that $\int \varphi L_1 \varphi = \int \varphi L_0 \varphi - \frac{2\left(\int \varphi \cdot \omega^2\right)\left(\int \varphi \omega\right)}{\int \omega^2} + \frac{\int \omega^3}{\left(\int \omega^2\right)^2} \cdot \left(\int \omega \varphi\right)^2$

$$\text{(c)} \Rightarrow \lambda_R \left[\int \varphi_R^2 + \int \varphi_I^2\right] = A + \left(\frac{2}{\int \omega^2} - \frac{\gamma}{\int \omega^2}\right) B - \frac{\int \omega^3}{\left(\int \omega^2\right)^2} C$$

where $A = \int \varphi_R L_1 \varphi_R + \int \varphi_I L_1 \varphi_I$

$$B = \left(\int \varphi_R \omega\right)\left(\int \varphi_R \omega^2\right) + \left(\int \varphi_I \omega\right)\left(\int \varphi_I \omega^2\right)$$

$$C = \left(\int \varphi_R \omega\right)^2 + \left(\int \varphi_I \omega\right)^2$$

Next take $\left(\int (a)\, \omega\right)\left(\int \omega\, \varphi_R\right) + \left(\int (b)\, \omega\right)\left(\int \omega\, \varphi_I\right)$;

also note that $\int \omega L_0 \varphi = \int \varphi L \omega = \int \varphi\, \omega^2$

so we get:

(d) $\lambda_R\, C = -\gamma \frac{\int \omega^3}{\int \omega^2} C + B$

(C) ~~and~~ (d) yields:

$$\lambda_R \left(\int \varphi_R^2 + \int \varphi_I^2\right) = A + C\left\{ - \frac{\int \omega^3}{\left(\int \omega^2\right)^2} + \left(\lambda_R + \gamma \frac{\int \omega^3}{\int \omega^2}\right)\left(\frac{2}{\int \omega^2} - \frac{\gamma}{\int \omega^2}\right)\right\}$$

or: $$\lambda_R\left[\left(\int \varphi_R^2 + \int \varphi_I^2\right) - \frac{C}{\int \omega^2}(2-\gamma)\right] = A - \frac{C \int \omega^3}{\left(\int \omega^2\right)^2}(1-\gamma)^2$$

Finally, decompose $\varphi_R = a_R\, \omega + \varphi_R^\perp$ where $\varphi_R^\perp \perp \omega$

$\varphi_I = a_I\, \omega + \varphi_I^\perp$ $\quad \varphi_I^\perp \perp \omega$

then $\int \varphi_I^2 + \int \varphi_R^2 = \left(a_R^2 + a_I^2\right)\int \omega^2 + \int \varphi_I^{\perp 2} + \varphi_R^{\perp 2}$

$$\frac{C}{\int \omega^2} = \left(a_R^2 + a_I^2\right)\int \omega^2$$

$$\Rightarrow \quad \lambda_R\left[\left(a_R^2 + a_I^2\right)\left(\int \omega^2\right)(\gamma - 1) + \left(\int \varphi_I^{\perp 2} + \int \varphi_R^{\perp 2}\right)\right] = A - (1-\gamma)^2 \frac{C \int \omega^3}{\left(\int \omega^2\right)^2}$$

We have $C \cancel{\geq} 0$ and by Lemma 2, $A<0$

Thus:

$$\lambda_R \left[(a_R^2 + a_I^2)\left(\int \omega^2\right)(\gamma - 1) + \int {\varphi_R^{\perp}}^2 + \int {\varphi_I^{\perp}}^2 \right] \leq 0$$

In particular, $\boxed{\lambda_R \leq 0 \text{ if } \gamma > 1}$.

Finally if $\lambda_R = 0$ then $A = (1-\gamma)^2 \, C \, \dfrac{\int \omega^3}{(\int \omega^2)^2}$.

Now $A \leq 0$ and $C \geq 0$ so either

$1 = \gamma$ or else $A = 0$ and $C = 0$.

But $A = 0 \Rightarrow \varphi_i, \varphi_R \in \text{span}\{\omega, \omega'\}$

and $C = 0 \Rightarrow \varphi_i, \varphi_R \perp \omega$

So either $\gamma = 1$ or else $\varphi = \omega'$, $\lambda_i = 0$

$\blacksquare$

Reference: J. Wei, "On single interior spike solutions of the Gierer-Meinhardt system: uniqueness and spectrum estimates", European J. Appl. Math, 1999 Vol 10 no. 4, 353-378.