Wigner's Circle law

- Random matrix model: $A_{ij} = \pm 1, \; i \neq j$

$A_{ij} = A_{ji}$

$A_{ii} = 0$

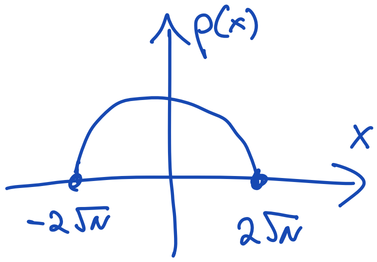
A is $N \times N$, with $N \to \infty$.

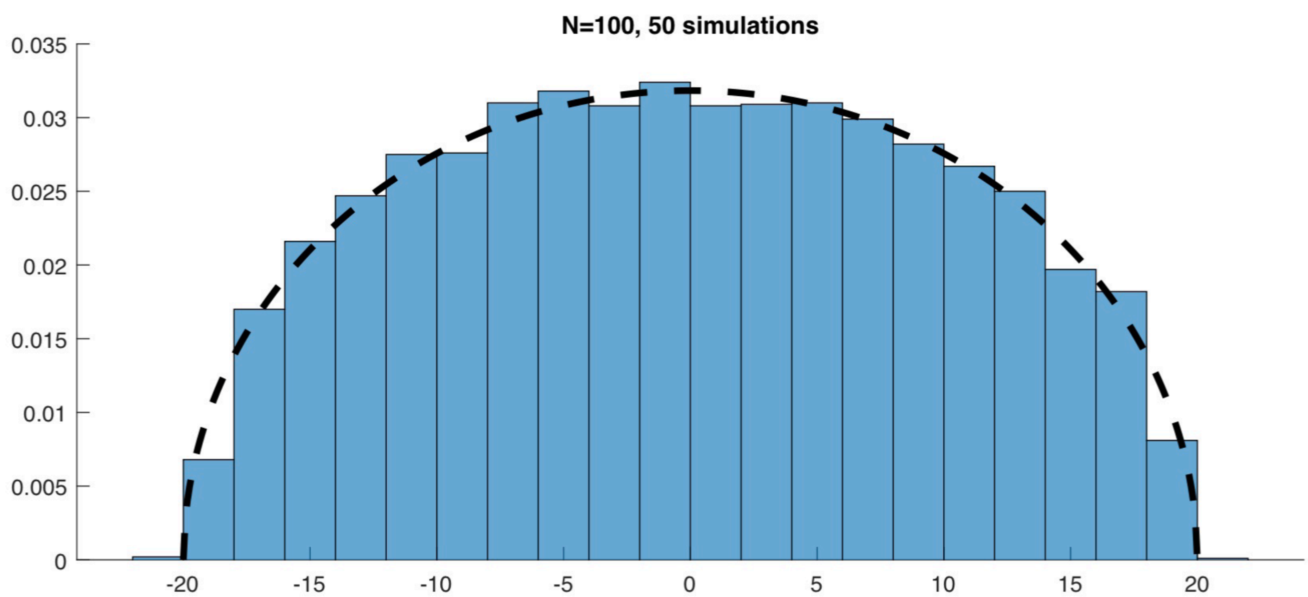
Ex

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

(1957) Wigner Circle law: In the limit $N \to \infty$, the distribution of eigenvalues of A is given by:

$$\rho(x) \sim C \begin{cases} 0, & |x| > R \\ (R^2 - x^2)^{\frac{1}{2}}, & |x| < R \end{cases} \quad \text{with } R = 2\sqrt{N}.$$





Idea of the derivation: $\mathrm{trace}(A^{\eta}) = \sum_{n} \lambda_j^{\eta}$

- So compute $\mathrm{trace}(A^{\eta})$.
- Density is $\rho(x) = \sum \delta(x - \lambda_j)$.
- Then $\int x^{\eta} \rho(x)\, dx = \sum \lambda_j^{\eta} = \mathrm{trace}(A^{\eta})$

  - This gives eq'n for $\rho(x)$.

Let $\varphi(\eta) = E\left[\left(A^{\eta}\right)_{1,1}\right]$. Then:

$$\int x^{\eta} \rho(x)\, dx = N \varphi(\eta). \qquad (*)$$

So: Task ①: compute $\varphi(\eta)$. Task ②: invert $(*)$.

Task 1 : compute $\varphi(r)$.

- There are $R = 2^{N\times(N-1)/2}$ possible matrices of this form;

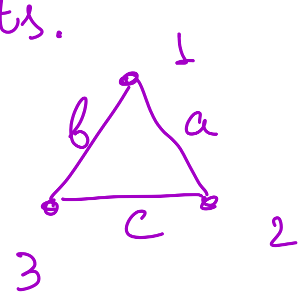
$(A^r)_{(1,1)}$

So

$$\varphi(r) = \frac{1}{R} \sum_{\text{all } A's} \sum_{i_1,\dots,i_{r-1}\in\{1..N\}} a_{1,i_1} a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_{r-1},1}$$

Graphical interpretation: $1\dots N$ are vertices; a_{ij} are edge weights; $(A^r)_{[i,j]}$ counts the number of paths from i to j, each weighted by the product of corresponding edge weights.

Ex:
$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$$



$$A^2 = \begin{bmatrix} a^2+b^2 & bc & ac \\ bc & a^2+c^2 & ab \\ ac & ab & b^2+c^2 \end{bmatrix}$$

$1\to 2\to 1 : a^2$
$1\to 3\to 1 : b^2$

- $$\varphi(r) = \frac{1}{R}\sum_A \sum_{\text{loops}} = \frac{1}{R}\sum_A \sum_{\text{"good loops"}}$$

"Good loop": is a loop that adds $+1$ regardless of A

Ex: • loop $1\to 2\to 2\to 1$ is bad

since it corresponds to product $A_{12} \underbrace{A_{22}}_{=0} A_{21} = 0 \quad \forall A$

- loop $1 \to 2 \to 3 \to 1$ is bad since

$$a_{12} a_{23} a_{31} = \pm 1$$

[half of A's have $a_{12} = +1$
another half $a_{12} = -1$]

- Good loop : $1 \to 2 \to 1$: $a_{12} a_{21} = 1 \ \forall A$

Def'n: Good loop : • both $i \to j$ and $j \to i$ must be present
• can't have $i \to i$

Then $\boxed{\varphi(r) = \#\text{ good loops of } r \text{ steps from } 1 \text{ to } 1}$

Computing good loops :

$$\varphi(0) = 1 \qquad \varphi(1) = 0,$$

$\varphi(2)$: $1 \to i \to 1 \quad \to \varphi(2) = N-1 \sim N$

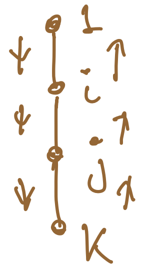
$\varphi(3) = 0, \ \varphi(\text{any odd}) = 0$

$\varphi(4)$: $1 \to i \to 1 \to j \to 1 \quad \sim N^2$ ways

$1 \to i \to j \to i \to 1 \quad \sim N^2$ ways

$$\varphi(4) \sim N^2 \cdot 2$$

$\varphi(6)$:



- $\downarrow\uparrow\,\boxed{(4)} \to N\,\varphi(4)$ $\qquad 1-i-1-\boxed{x-y-x-1}$
- $\downarrow\boxed{\downarrow\uparrow}\uparrow\boxed{(2)} \to N\,\boxed{\varphi(2)}\,\boxed{\varphi(2)}$ $\qquad 1-\boxed{i-j-i}-1-\boxed{x-1}$
- $\downarrow\boxed{\downarrow\downarrow\uparrow\uparrow}\uparrow \to N\,\boxed{\varphi(4)}$ $\qquad 1-\boxed{i-j-k-j-i}-1$

$$\varphi(6) \sim N\left[\varphi(0)\varphi(4) + \varphi(2)\varphi(2) + \varphi(4)\varphi(0)\right]$$
$$\sim N^3 \cdot 5$$

In general: $\displaystyle \varphi(r+2) \sim N \sum_{\substack{j=2 \\ j \text{ even}}}^{r} \varphi(j)\,\varphi(r-j)$

<u>Scale out N</u>: $\varphi(r) \sim N^{\frac{r}{2}}\, t_r$ where:

t_r is the number of loops with r steps, starting from A, on the "half-line" graph,

$$\boxed{t_{r+2} = \sum_{\substack{j=2 \\ j \text{ even}}}^{r} t_j\, t_{r-j}}$$

- $t_0 = 1$, $t_2 = 1$, $t_4 = 2$, $t_6 = 5$, $t_8 = 14$

These are called <u>Catalan numbers</u>!

<u>Generating function</u>:

Let $T(x) = t_0 + t_2 x^2 + t_4 x^4 + \ldots$

$$T^2 = \underbrace{t_0^2}_{\underset{t_2}{\parallel}\ 1} + (\underbrace{t_0 t_2 + t_2 t_0}_{t_4}) x^2 + (\underbrace{t_0 t_4 + t_2 t_2 + t_4 t_0}_{t_6}) x^4 + \dots$$

$$= \frac{T-1}{x^2} \Rightarrow \boxed{T^2 x^2 - T + 1 = 0}$$

$$\Rightarrow \boxed{T(x) = \frac{1 - \sqrt{1-4x^2}}{2x^2}} \quad (*)$$

It follows from Taylor-expansion of (*)

that $t_{2s} = \frac{1}{s}\binom{2s}{s}$ (**)

(which is an explicit formula for Catalan numbers)

Summary so far: $\int x^q \rho(x)\, dx = N^{\frac{q}{2}+1} t_q$

Task 2: Determine $\rho(x)$.

- Wigner used (**) in his derivation explicitly; here I will show a simpler derivation which uses (*) directly.

- Cauchy formula: if $f(z) = \sum a_n z^n$, then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz,$$ where C includes 0 but excludes any singularities of f.

$$t_n = \frac{1}{2\pi i} \int_{|z|=\frac{1}{2}} \frac{1-\sqrt{1-4z^2}}{2z^{n+3}}\, dz \quad , \quad z = \frac{1}{2} e^{i\theta}$$

$$= \frac{-1}{\pi} 2^n \int_{-\pi}^{\pi} e^{-i\theta(n+2)} \left(1 - e^{2\theta i}\right)^{\frac{1}{2}} d\theta$$

$$= -\frac{2^{n+1}}{\pi} \int_{-\pi/2}^{\pi/2} \underbrace{e^{i(n+2)\theta} \left(1 - e^{-2\theta i}\right)^{\frac{1}{2}}}_{e^{i(n+1)\theta}\left(e^{2\theta i} - 1\right)^{\frac{1}{2}}} d\theta .$$

$$u = e^{i\theta}$$

$$= \frac{2^{n+1}}{\pi} \int_{-1}^{1} u^n \left(1-u^2\right)^{\frac{1}{2}} du$$

<u>Recall:</u> $\int x^r \rho(x)\, dx = N^{\frac{r}{2}+1} t_r$

$$\int x^r \rho(x)\, dx = N^{r/2+1} \frac{2^{r+1}}{\pi} \int_{-1}^{1} u^r (1-u^2)^{\frac{1}{2}} du$$

$$= C\alpha^r \int_{-1}^{1} u^r (1-u^2)^{\frac{1}{2}} du$$

$$\left(\text{where } \alpha = 2N^{\frac{1}{2}}, \quad C = \frac{2N}{\pi}\right)$$

$u = \frac{x}{\alpha}$: $\quad \therefore = \frac{C}{\alpha} \int_{-\alpha}^{\alpha} x^r \left(1 - \frac{x^2}{\alpha^2}\right)^{\frac{1}{2}} dx$

$$= \frac{C}{\alpha^2} \int_{-\alpha}^{\alpha} x^r \left(\alpha^2 - x^2\right)^{\frac{1}{2}} dx$$

$$\Rightarrow \boxed{\rho(x) = \frac{2}{\alpha^2 \pi} \begin{cases} (\alpha^2 - x^2)^{\frac{1}{2}}, & |x| < \alpha \\ 0, & |x| > \alpha, \end{cases} \qquad \alpha = 2N^{\frac{1}{2}}}$$