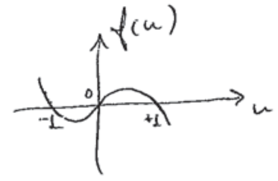
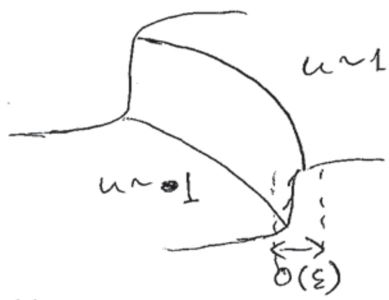
Travelling wave inside a thin channel:

Consider the Allen-Cahn model of phase separation:

$$(1)\quad \begin{cases} u_t = \varepsilon^2 \Delta u + f(u), & x \in \Omega \subset \mathbb{R}^2 \\ \partial_n u = 0, & x \in \partial\Omega \\ f(u) = -2u(u-1)(u+1) \end{cases}$$

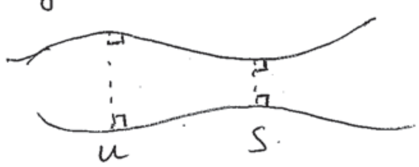


- In the limit $\varepsilon \to 0$ the solution quickly forms an interface layer of thickness $O(\varepsilon)$, with $u \sim -1$ on one side of the interface and $u \sim 1$ on the other side:



- Once formed, the interface will move slowly (speed $O(\varepsilon^2)$) according to the mean curvature law: the the speed in the direction orthogonal to the interface is proportional to the curvature of the interface at that point:

- If Ω is a bounded domain, the interface will move in such a way as to minimize its length.
- The steady state is a straight interface; orthogonal to $\partial\Omega$:

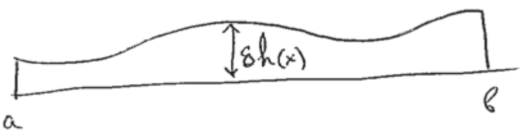


It is stable at the "neck" of the domain, unstable at the "hips".

We consider the case of <u>thin domain</u>:

$$\Omega = \{ (x,y) : 0 \le y \le \delta h(x),\ a \le x \le b \}$$

where $\delta \ll \varepsilon \ll 1$, $h(x) > 0$.



Then (1) can be reduced to a one-dimensional problem as follows:

<u>Change variables</u>: $z = \frac{y}{\delta h(x)}$, $u(x,y) = U(x,z)$.

Then
$$u_x = U_x - \frac{y h'}{\delta h^2} U_z = U_x - z \frac{h'}{h} U_z$$
$$u_{xx} = U_{xx} - 2z \frac{h'}{h} U_{xz} + \frac{h'^2}{h^2}\left[z^2 U_{zz} + 2z U_z\right] - \frac{h''}{h^2} z U_z$$
$$u_y = \frac{1}{\delta h} U_z$$
$$u_{yy} = \frac{1}{\delta^2 h^2} U_{zz}$$

The Boundary conditions are:
$$\begin{cases} u_x h' \delta - u_y = 0 & \text{at } y = \delta h(x) \\ u_y = 0 & \text{at } y = 0 \end{cases}$$

They become:
$$\begin{cases} \left(U_x - z\frac{h'}{h} U_z\right) h' \delta^2 = \frac{U_z}{h}, & \text{at } z = 1 \\ U_z = 0 & \text{at } z = 0 \end{cases}$$

<u>Expand</u> $U = U_0(x,z) + \delta^2 U_1(x,z) + \dots$

<u>$O(1)$</u>:
$$\begin{cases} U_{0zz} = 0, & z \in (0,1) \\ U_{0z} = 0 & \text{at } z = 0, 1 \end{cases} \Rightarrow \boxed{U_0(x,z) = U_0(x)}$$

$O(\delta^2)$:
$$\begin{cases} U_{0t} = \varepsilon^2 \left(U_{0xx} + \frac{1}{h^2} U_{1zz} \right) + f(U_0) & (*) \\ U_{1z} = h h' U_{0x} & \text{at } z = 1 \\ U_{1z} = 0 & \text{at } z = 0 \end{cases}$$

Integrate (*) w.r.t. z on $z \in [0,1]$:

$$u_{0t} = \varepsilon^2 \left(u_{0xx} + \frac{1}{h^2} \left(\underbrace{u_{1z}\Big|_0^1}_{h'h\, u_{0x}} \right) \right) + f(u_0)$$

$$\Rightarrow \quad u_{0t} = \varepsilon^2 \left(u_{0xx} + \frac{h'(x)}{h(x)} u_{0x} \right) + f(u_0) \qquad (2)$$

Eq'n (2) is the <u>homogenized</u> version of (1) for thin domains, (2) is 1-D pbm.

Drop $u_0 \to u$, our <u>starting point</u> is:

$$(3) \quad \boxed{\begin{cases} u_t = \varepsilon^2 \left(u_{xx} + \frac{h'}{h} u_x \right) + f(u), & a < x < b \\ u'(a) = 0 = u'(b) \end{cases}}$$

<u>Anzatz</u>: $x = x_0(\varepsilon^2 t) + \varepsilon y$; $u = u_0(y) + \varepsilon u_1 + \dots$

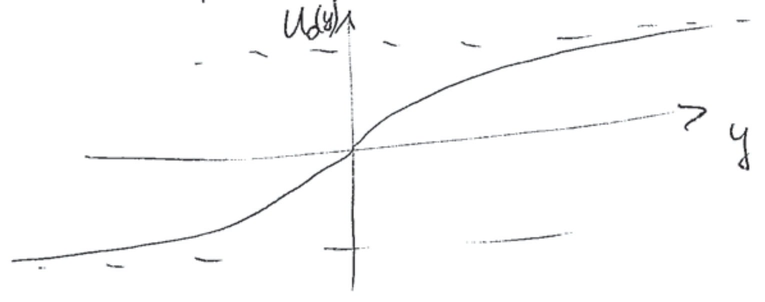
Then $u_x = \frac{1}{\varepsilon} u_y$, $u_{xx} = \frac{1}{\varepsilon^2} u_{yy}$, $\frac{h'(x)}{h(x)} \sim \frac{h'(x_0)}{h(x_0)} + O(\varepsilon)$

$u_t = -\varepsilon\, x_0'(\varepsilon^2 t)\, u_y$

$\Rightarrow O(1)$: $\quad u_{0yy} + f(u_0) = 0$

$$\Rightarrow u_0(y) = \pm \tanh(y)$$

[heterodinic orbit that describes the shape of interface]



$\underline{O(\varepsilon)}$: $\int_{-\infty}^{\infty} U_{0y}\left(-x_0' U_{0y} = U_{1yy} + U_1 f'(U_0) + \frac{h'(x_0)}{h(x_0)} U_{0y}\right)$

Integrate by parts & use: $(U_{0y})_{yy} + f'(U_0) U_{0y} = 0$:

$$-x_0'(\varepsilon t) \int U_{0y}^2 = \frac{h'(x_0)}{h(x_0)} \int U_{0y}^2$$

$$\Rightarrow \boxed{\frac{dx_0}{dt} = -\varepsilon^2 \frac{h'(x_0)}{h(x_0)}} \quad (4)$$

- The steady states of ODE (4) correspond to max/min of h;
- It is stable if $\left(\frac{h'}{h}\right)' > 0 \Leftrightarrow x_0$ is a min of $h(x)$
- Unstable if x_0 is a max of $h(x)$
- Conclusion: Travelling wave gets "stuck" at the "neck" of the domain (where h has a min).

Numerical example:

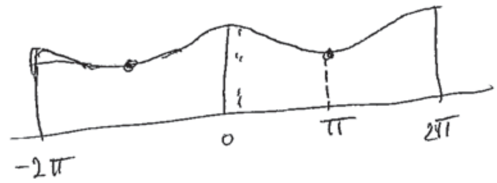
Note that $u(x,t) \sim \pm \tanh\left(\frac{x - x_0(t)}{\varepsilon}\right)$ where

$x_0(t)$ evolves according to (4).

Take $h(x) = 1 + \frac{\cos x}{2}$, $x \in [-2\pi, 2\pi]$

and take i.c.

$$u(x,0) = \tanh\left(\frac{x - 0.1}{\varepsilon}\right)$$

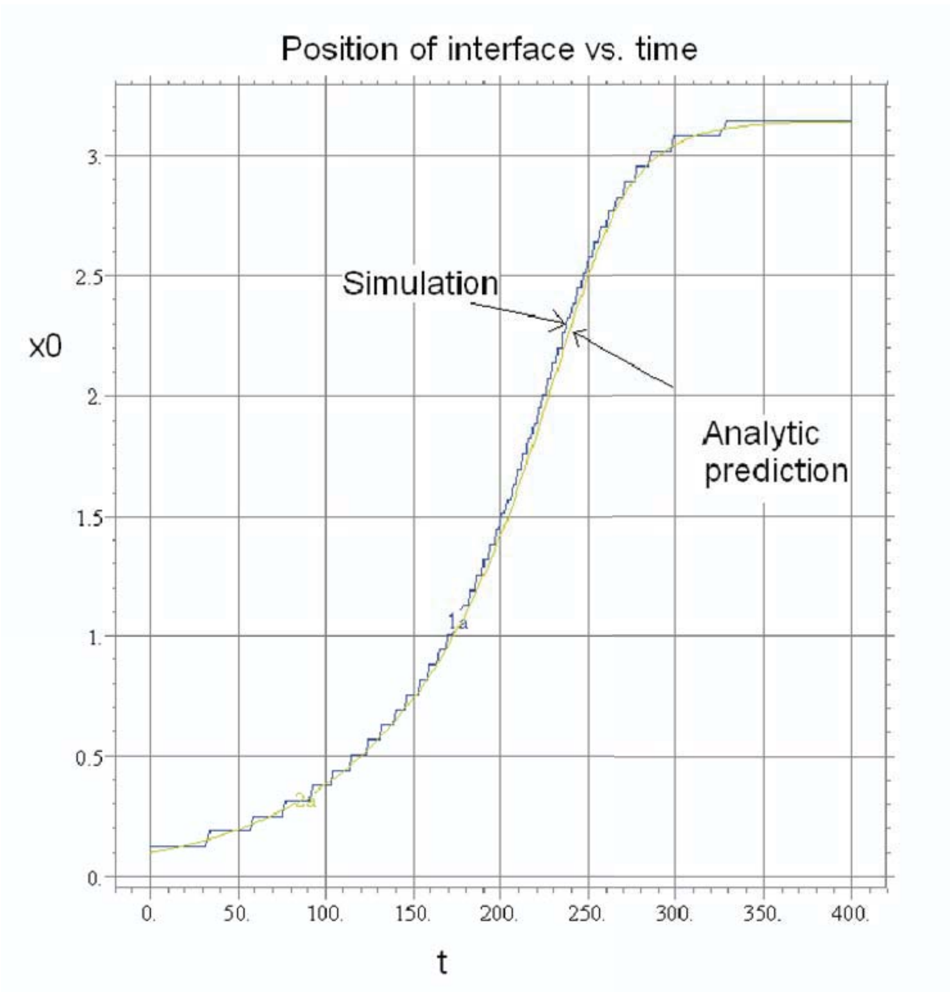


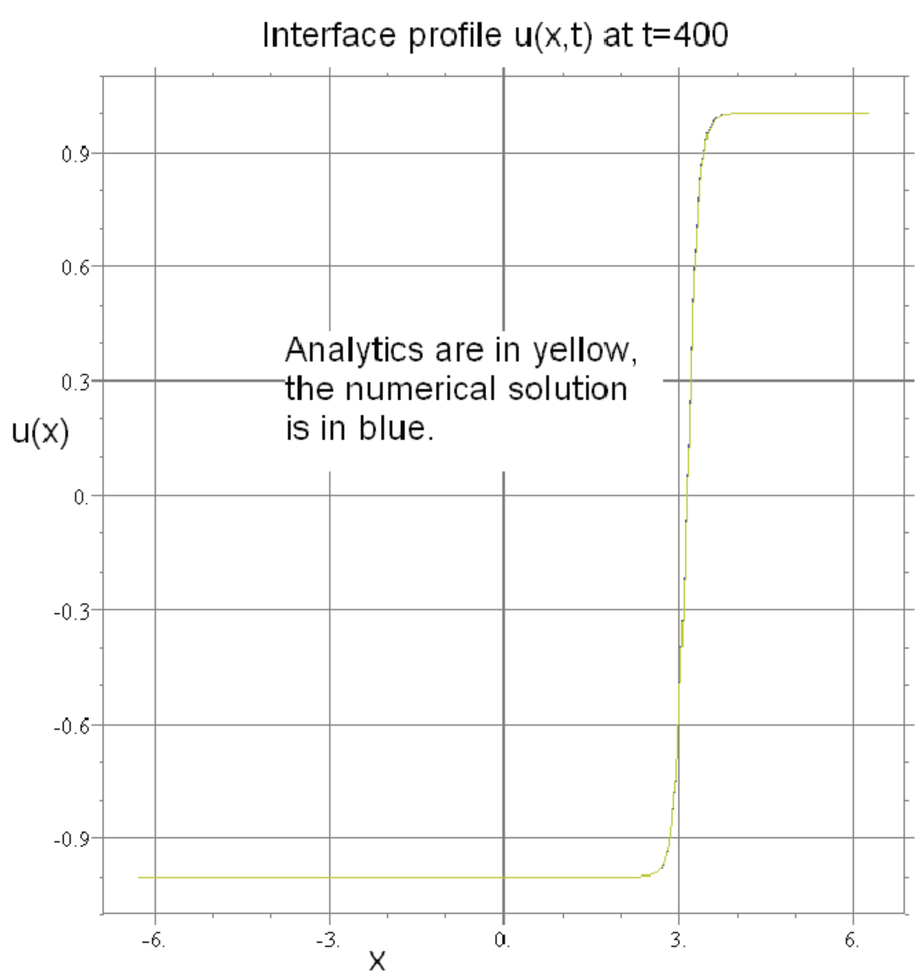
Take $\varepsilon = 0.2$; using the software "FlexPDE" we get the results shown on the next page. The agreement is very good. as expected,

$x_0 \to \pi$ as $t \to \infty$

References:

- S. Allen and J.W. Cahn, "A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening", Acta. Metall. 27 (1979), 1084–1095
- T. Rubinstein, P. Sternberg and J.B. Keller, "Fast reaction, slow diffusion, and curve shortening", SIAM J. Appl. Math., Vol. 49 no. 1 (1989), 116–133.


Position of interface vs. time
x0
t
Simulation
Analytic prediction


Interface profile u(x,t) at t=400
u(x)
x
Analytics are in yellow, the numerical solution is in blue.

Further questions

1) Consider the Allen-Cahn system (1):

$$(1) \quad \begin{cases} u_t = \varepsilon^2 \Delta u + f(u), & x \in \Omega, \quad f(u) = -2u(u^2-1) \\ \partial_n u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a <u>thin tube</u>, defined as:

$$\Omega = \{ (x,y,z) : (y,z) \in \delta D(x), \ x \in (a,b) \}$$

where $\delta \ll \varepsilon \ll 1$ and $D(x) \subset \mathbb{R}^2$ is a 2-d domain that varies smoothly with x.

Show that in the limit $\delta \ll \varepsilon \ll 1$, the system (1) is approximated by

$$\begin{cases} u_t = \varepsilon^2 \frac{1}{A(x)} \left(u'(x) A(x) \right)' + f(u) \,, & x \in (a,b) \\ u'(a) = u'(b) = 0 \end{cases}$$

where $A(x) = \text{area}(D(x))$.

2) Consider $u_t = \varepsilon^2 \Delta u + f(u) + \varepsilon g(u)$, $x \in \mathbb{R}^2$.

where $f(u) = -2u(u-1)(u+1)$

and $\int_{-1}^{1} g(u) = 1$.

a) Find a radially symmetric travelling wave solution of the form:

$$u(x) = U\left(\frac{r - r_0(\varepsilon^2 t)}{\varepsilon} \right), \quad r = |x|.$$

Determine the ODE for r_0

b) Does the ODE you found in part (a) have a steady state? Is it stable?