MATH 5230/4230, Homework 2

Due: Thursday, 7 Feb

1. Let μ be the principal eigenvalue to

$$-\Delta\phi = \mu\phi \text{ inside } \Omega'$$
$$\phi = 0 \text{ on } \partial\Omega'$$

Given Ω, we need to choose Ω' with $\Omega \subset \Omega'$ such that

$$\frac{\min_{\Omega}\phi}{\max_{\Omega'}\phi} \geq 1/3.$$

   (a) Suppose that $\Omega = [-L, L] \subset \mathbb{R}$. How should you choose Ω' and what would the corresponding μ be?

   (b) [BONUS] Suppose that $\Omega \subset B_L \subset \mathbb{R}^2$ where B_L is the unit disk in two dimensions centered at zero. How should you choose Ω' and what would the corresponding μ be?

2. Consider the following problem:

$$\begin{cases} u_{rr} + \frac{n-1}{r}u_r + \lambda e^u = 0 \\ u'(0) = 0, \quad u(1) = 0, \quad u > 0 \text{ inside } [0,1) \end{cases} . \tag{1}$$

   (a) Using a computer, draw the bifurcation diagram of $u(0)$ vs. λ for $n = 1, 2, 3$. You should observe that (1) has infinitely many solutions if $n = 3$ and $\lambda = 2$ but at most two solutions if $n = 1, 2$. Also observe that there exists λ^* such that solution to (1) exists if and only if $\lambda \leq \lambda^*$. HINT: it may be useful to change variables $v = e^u$ first.

   (b) Prove the existence of λ^* and give an upper bound for it.

   (c) Using the method of sub/supersolutions, give a lower bound for λ^*.

   (d) In one dimension, $n = 1$, this problem can be solved explicitly. Do it and explicitly compute λ^* for $n = 1$. Does it agree with upper/lower bounds you found in parts (b) and (c)?

   (e) Use an appropriate change of variables and a phase-plane/stability analysis to study the bifurcation diagrams you sketched in (a).

   (f) The bifurcation diagram changes qualitatively again if $n > n^* > 3$. Determine this n^* and draw the bifurcation diagram for $n > n^*$.

3. Consider the Gierer-Meinhardt reaction-diffusion system

$$a_t = D_1 a_{xx} - a + a^2/h, \qquad \tau h_t = D_2 h_{xx} - h + a^2.$$

   (a) Suppose $D_1 = D_2 = 0$. Under what conditions on τ is the steady state $a = 1, h = 1$ stable?

   (b) Under conditions found in part (a), perform the Turing stability analysis of the steady state $a = h = 1$ for general $D_1, D_2 > 0$. Compute a critical threshold D_{1c} such that the homogeneous state is unstable for $D_1 < D_{1c}$ and is stable for $D_1 > D_{1c}$.

   (c) Use FlexPDE to compute the solution and validate the result you found in part (b). Choose parameters appropriately to demonstrate how the instability happens as D_1 crosses the critical threshold. See course website for sample FlexPDE code. Print out all the relevant figures.