MATH 5230/4230, Homework 3

due date: Thursday Oct 23rd.

1. Let μ be the principal eigenvalue to

$$-\Delta\phi = \mu\phi \text{ inside } \Omega'$$
$$\phi = 0 \text{ on } \partial\Omega'$$

Given a constant a, we need to choose Ω' with $\Omega \subset \Omega'$ such that

$$\frac{\min_{\Omega}\phi}{\max_{\Omega'}\phi} \geq a.$$

   (a) Suppose that $\Omega = [-L, L] \subset \mathbb{R}$. How should you choose Ω' and what would the corresponding μ be?

   (b) [BONUS] Suppose that $\Omega \subset B_L \subset \mathbb{R}^2$ where B_L is the unit disk in two dimensions centered at zero. How should you choose Ω' and what would the corresponding μ be?

2. Consider the following problem:

$$\begin{cases} u_{rr} + \frac{n-1}{r}u_r + \lambda e^u = 0 \\ u'(0) = 0, \quad u(1) = 0, \quad u > 0 \text{ inside } [0,1) \end{cases}. \tag{1}$$

   (a) Reformulate this boundary value probelm as an initial value problem. HINT: it may be useful to change variables $v = e^u$.

   (b) Using a computer, draw the bifurcation diagram of $u(0)$ vs. λ for $n = 1, 2, 3$. You should observe that (1) has infinitely many solutions if $n = 3$ and $\lambda = 2$ but at most two solutions if $n = 1, 2$. Also observe that there exists λ^* such that solution to (1) exists if and only if $\lambda \leq \lambda^*$.

   (c) Prove the existence of λ^* and give an upper bound for it.

   (d) Using the method of sub/supersolutions, give a lower bound for λ^*.

   (e) In one dimension, $n = 1$, this problem can be solved explicitly. Do it and explicitly compute λ^* for $n = 1$. Does it agree with upper/lower bounds you found in parts (b) and (c)?

   (f) Use an appropriate change of variables and a phase-plane/stability analysis to study the bifurcation diagrams you sketched in (a).

   (g) The bifurcation diagram changes qualitatively again if $n > n^* > 3$. Determine this n^* and draw the bifurcation diagram for $n > n^*$.

3.