MATH 5230/4230, Homework 3

1. Consider the following problem:

$$\begin{cases} u_{rr} + \frac{n-1}{r} u_r + \lambda e^u = 0 \\ u'(0) = 0, \quad u(1) = 0, \quad u > 0 \text{ inside } [0,1) \end{cases} . \tag{1}$$

   (a) Change the variables $v = e^u$.

   (b) Reformulate the boundary value probelm (1) as an initial value problem for v.

   (c) Solve the problem in (b) numerically using a computer, and draw the bifurcation diagram of $u(0)$ vs. λ for $n = 1, 2, 3$. See website for Maple code to do this for MEMS problem. You should observe that (1) has infinitely many solutions if $n = 3$ and $\lambda = 2$ but at most two solutions if $n = 1, 2$. Also observe that there exists λ^* such that solution to (1) exists if and only if $\lambda \leq \lambda^*$.

   (d) Prove the existence of λ^* and give an upper bound for it.

   (e) Use an appropriate change of variables and a phase-plane/stability analysis to study the bifurcation diagrams you sketched in (c).

   (f) The bifurcation diagram changes qualitatively again if $n > n^* > 3$. Determine this n^* and draw the bifurcation diagram for $n > n^*$.

   (g) Using the method of sub/supersolutions, give a lower bound for λ^* (see also question 2).

   (h) In one dimension $n = 1$, this problem can be solved explicitly. Do it and explicitly compute λ^* for $n = 1$. Does it agree with upper/lower bounds you found in parts (b) and (g)?

2. In class, we encountered the following problem. Let μ be the principal eigenvalue to

$$-\Delta\phi = \mu\phi \text{ inside } \Omega'$$
$$\phi = 0 \text{ on } \partial\Omega'$$

   Given a constant a, we need to choose Ω' with $\Omega \subset \Omega'$ such that

$$\frac{\min_{\Omega} \phi}{\max_{\Omega'} \phi} \geq a.$$

   (a) Suppose that $\Omega = [-L, L] \subset \mathbb{R}$. How should you choose Ω' and what would the corresponding μ be?

   (b) Suppose that $\Omega \subset \mathbb{R}^2$. How should you choose Ω' and what would the corresponding μ be? Hint: use question 2 to replace the domain by an appropriate square domain. You may use the fact that enlarging domain decreases the principal eigenvalue.