MATH 5230/4230, Homework 4

Due: Thurs, Nov 6

1.

1) What is the optimal location for organic farm?

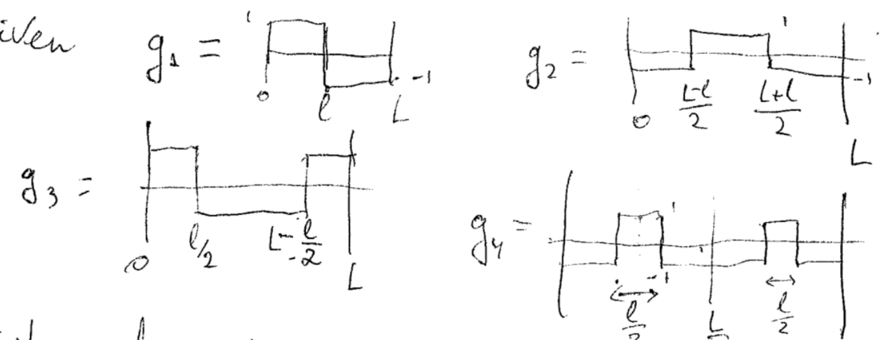
a) Consider the problem

$$\lambda \varphi = \varphi'' + g(x)\varphi, \quad \varphi'(0) = 0 = \varphi'(L) \qquad (Q1)$$

where $g(x) = \begin{cases} 1, & x \in (a, a+\ell) \\ -1, & x \in [0, L] \setminus (a, a+\ell) \end{cases}$

Show that λ is minimized when $a = \frac{L-\ell}{2}$, i.e. when the organic farm is located in the middle of the interval.

b) Given



Which configuration minimizes λ?

2. Consider the eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u = 0, &\quad \text{inside } B_1(0) \subset \mathbb{R}^2 \\ \partial_n u = 0, &\quad \text{on } \partial B_1(0) \\ u = 0, &\quad \text{on } \partial B_\varepsilon(\xi) \end{aligned} \qquad (1)$$

Here, ε is assumed small and $B_\varepsilon(\xi) \subset B_1(0)$.

(a) First, suppose that $\xi = 0$ so that the problem is radially symmetric. In this case, find an implicit expression for λ in terms of Bessel J_0 and Y_0 functions. See Wikipedia or ask me if you need more info on Bessel functions.

(b) Using the following expansions for Bessel J_0 and Y_0 functions of small arguments,

$$\begin{aligned} J_0(z) &\sim \frac{2}{\pi}\ln(z) + \frac{2}{\pi}(\gamma - \ln 2) \text{ as } z \to 0, \\ Y_0(z) &\sim 1 + O(z^2) \text{ as } z \to 0, \end{aligned}$$

where $\gamma = 0.577...$ is the Euler constant, find the asymptotic formula for λ in the limit $\varepsilon \to 0$.

(c) Use Maple to compute λ as determined by (a). Hint: the command `fsolve` will be useful here: for example `fsolve(x^2=2,x=1.4);` uses fsolve to solve for $\sqrt{2}$, the second argument provides an initial guess. Then compare with the asymptotic formula for λ that you obtained in part (b). Do this for two values, $\varepsilon = 0.1$ and $\varepsilon = 0.05$. Comment on what you observe for the error behaviour.

(d) Now do the case of general ξ in (1). Here are some steps:

- Expand $\lambda = \delta\lambda_0 + \delta^2\lambda_1 + \ldots$, and $u(x) = u_0 + \delta u_1(x) + \ldots$, where $\delta := \frac{1}{\log(1/\varepsilon)} \ll 1$ and u_0 is constant.
- You will find that $u_1 \sim AG(x, \xi)$ where A is some constant that you will need to determine and G is the same Neumann's Green's function that we saw in class.
- Determine λ_0. Make sure to double-check that whever you got agrees with part (b) when $\xi = 0$. How does the answer depend on ξ?
- BONUS: Determine λ_1. Then compare with what you got in part (b).