MATH 5230/4230, Homework 4

1. In class, we considered the following model of bacterial motion/aggregation:

$$\frac{dU_j}{dt} = a(U_{j-1} + U_{j+1}) - (2a+c)U_j + c\left(U_{j-1}\frac{U_j}{U_{j-2}+U_j} + U_{j+1}\frac{U_j}{U_{j+2}+U_j}\right) \tag{1}$$

Here, a is the rate with which the bacteria changes direction at random from left to right as well as from right to left, and c is the rate with which the bacteria changes the direction towards its neighbours.

(a) Derive the continuum model (1) by taking the limit of lattice spacing $h \to 0$. That is, let $U_j(t) = u(x,t)$ so that $U_{j+k}(t) = u(x+kh,t)$. Show that up to $O(h^2)$ terms, you get

$$u_t = h^2\left(a - \frac{c}{2}\right)u_{xx}$$

Conclude that the aggregation forms if and only if $c/2 > a$.

(b) Test your prediction by using numerical simulations of (1) with well chosen parameters. Be sure to illustrate both stable and unstable cases. Some matlab code is available on the website for this.

(c) In the case $c/2 > a$, expand to a higher order and *verify* that you obtain the PDE of the form

$$u_t = -Ah^2u_{xx} - Bh^4u_{xxxx} + Ch^4\left(\frac{u_x u_{xx}}{u}\right)_x \tag{2}$$

where $A = \frac{c}{2} - a$, $B = \frac{1}{12}\left(\frac{7}{2}c - a\right)$ and $C = c/2$.

(d) Now suppose that the random part of the motion has a bias as well. In other words, the bacteria changes direction from left to right at rate a_l and changes direction from right to left with a possibly different rate a_r. Modify equation (1) to account for this. Note: when $a_l = a_r$, your new equation should reduce to (1) , with $a = a_r = a_l$ in this case.

(e) Repeat parts (a,b,c) for the model you derived in part (d).

2. Consider the eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u &= 0, \quad \text{inside } B_1(0) \subset \mathbb{R}^2 \\ \partial_n u &= 0, \quad \text{on } \partial B_1(0) \\ u &= 0, \quad \text{on } \partial B_\varepsilon(\xi) \end{aligned} \tag{3}$$

Here, ε is assumed small and $B_\varepsilon(\xi) \subset B_1(0)$.

(a) First, suppose that $\xi = 0$ so that the problem is radially symmetric. In this case, find an implicit expression for λ in terms of Bessel J_0 and Y_0 functions. See Wikipedia or ask me if you need more info on Bessel functions.

(b) Using the following expansions for Bessel J_0 and Y_0 functions of small arguments,

$$\begin{aligned} Y_0(z) &\sim \frac{2}{\pi}\ln(z) + \frac{2}{\pi}(\gamma - \ln 2) \quad \text{as } z \to 0, \\ J_0(z) &\sim 1 + O(z^2) \quad \text{as } z \to 0, \end{aligned}$$

where $\gamma = 0.577...$ is the Euler constant, find the asymptotic formula for λ in the limit $\varepsilon \to 0$.

(c) Use Maple to compute λ as determined by (a). Hint: the command `fsolve` will be useful here: for example `fsolve(x^2=2,x=1.4);` uses fsolve to solve for $\sqrt{2}$, the second argument provides an initial guess. Then compare with the asymptotic formula for λ that you obtained in part (b). Do this for two values, $\varepsilon = 0.1$ and $\varepsilon = 0.05$. Comment on what you observe for the error behaviour.

(d) Now do the case of general ξ in (1). Here are some steps:

- Expand $\lambda = \delta\lambda_0 + \delta^2\lambda_1 + \ldots$, and $u(x) = u_0 + \delta u_1(x) + \ldots$, where $\delta := \frac{1}{\log(1/\varepsilon)} \ll 1$ and u_0 is constant.
- You will find that $u_1 \sim AG(x,\xi)$ where A is some constant that you will need to determine and G is the same Neumann's Green's function that we saw in class.
- Determine λ_0. Make sure to double-check that whever you got agrees with part (b) when $\xi = 0$. How does the answer depend on ξ?
- BONUS: Determine λ_1. Then compare with what you got in part (b).

3. Consider the Dirichlet Green's function on a square:

$$\Delta G = \delta(\vec{x} - \vec{x}_0), \quad \vec{x}, \vec{x}_0 \in D, \qquad G = 0 \text{ for } x \in \partial D \tag{4}$$

where $D = \{(x,y) : x \in (0,L),\ y \in (0,H)\}$. Let R be the regular part, that is,

$$R(\vec{x}, \vec{x}_0) = G(\vec{x}, \vec{x}_0) - \frac{1}{2\pi} \ln|\vec{x} - \vec{x}_0|.$$

The goal is to use the method we discussed in class to determine $R_0 = R(c,c)$ where $c = (L/2, H/2)$ is at the center of the square.

(a) Decompose $\delta(\vec{x} - \vec{x}_0) = \sum v_m(x)\phi_m(y)$ and $G = \sum G_m(x)\phi_m(y)$ where $\phi_m(y)$ are the appropriate eigenfunctions (hint: $\phi_m(y) = \sin(\textit{something})$).

(b) Find the solution to

$$u_{xx} - \lambda^2 u = \delta(x - x_0), \quad u(0) = 0 = u(L)$$

(c) Using (a) and (b), solve for G_m

(d) Use the resummation technique we saw in class to obtain an infinite series expansion for R_0

(e) Test your result by using the formula obtained in (d), compute R_0 for $(L,H) = (2,1)$ and $(L,H) = (1,2)$. By symmetry, these should be the same. Take enough terms to get the answer to 3 significant digits. NOTE: the two Comment on convergence.