MATH 5230/4230, Homework 4

Due: Thurs 7 March.

1. We consider Gray-Scott model,

$$
\begin{aligned}
u_t &= \varepsilon^2 u_{xx} - u + Au^2 v \\
0 &= v_{xx} - v + 1 - \frac{u^2 v}{\varepsilon}.
\end{aligned}
$$

Here, $0 < \varepsilon \ll 1$ and $A = O(1)$.

(a) Construct a symmetric steady state on the domain $x \in [0, L]$ with Neumann boundary conditions for u, v, consisting of a half-spike centered at the origin. That is, to leading order, the solution for u should be of the form $u(x) \sim \xi w(x/\varepsilon)$ where $w(y) = \frac{3}{2}\operatorname{sech}^2(y/2)$ is the ground-state solution to $w_{yy} - w + w^2 = 0$. Your goal is to determine the value the constant ξ as a function of A and L. Show that ξ satisfies a quadratic equation. It has two solutions $\xi_- < \xi_+$ if $A > A_0$ or no solutions if $A < A_0$. Determine the value of A_0.

(b) Formulate the nonlocal eigenvalue problem that determines the stability of the solutions found in part (a). For $A > A_0$, show that one of the solutions you found in part (a) is unstable while the other is stable.

(c) Consider the same eigenvalue problem as in part (b), but with boundary conditions for the eigenfunction of the form $\phi'(0) = 0$ and $\phi(L) = 0$. Both the steady state and the eigenvalue problem can be extended from $[0, L]$ to $[0, 2L]$, by reflecting at $x = L$. Then the steady state consists of two boundary spikes: half-spike at $x = 0$ and another half-spike at $x = 2L$. Show that there exists a critical spike radius L_c such that this configuration is stable if $L > L_c$ and is unstable if $L < L_c$. Determine L_c.

(d) Use FlexPDE to compare the result you found in part (c) with full numerical simulations of the GS model. Illustrate your result by showing both stable and unstable configurations. See sample script on the website.

(e) Determine the asymptotic equation of motion of a single spike whose center is at x_0, inside the domain $[-L, L]$. Compare your result with full numerics using FlexPDE.