1. In this problem you are asked to construct radially propagating interface in 2D to the PDE

$$u_t = \varepsilon^2 \Delta u - 2(u + \varepsilon a)(u-1)(u+1), \quad x \in \mathbb{R}^2.$$

where we assume that $0 < \varepsilon \ll 1$ and $a = O(1)$; we allow a to be either positive or negative.

(a) Find a travelling wave solution of the form

$$u(x) \sim U_0 \left(\frac{r - r_0(\varepsilon^2 t)}{\varepsilon} \right), \; r = |x|,$$

where $U_0(z) = \pm \tanh(z)$. Derive an ODE for r_0; note that you get two different answers depending on whether you take $+\tanh$ or $-\tanh$. HINT: $\Delta u = u_{rr} + \frac{1}{r} u_r$.

(b) Does the ODE you derived in part (a) have an equilibrium point? If so, is that equilibrium stable or unstable?

2. Consider the Schnakenberg model,

$$u_t = \varepsilon^2 u_{xx} - u + u^2 v$$
$$0 = v_{xx} + a - \frac{u^2 v}{\varepsilon}.$$

Here, $0 < \varepsilon \ll 1$ and $a = O(1)$.

(a) Construct a symmetric steady state on the domain $x \in [-L, L]$ with Neumann boundary conditions for u, v, consisting of a spike centered at the origin. That is, to leading order, the solution for u should be of the form $u(x) \sim \xi w(x/\varepsilon)$ where $w(y) = \frac{3}{2} \operatorname{sech}^2(y/2)$ is the ground-state solution to $w_{yy} - w + w^2 = 0$. Your goal is to determine the value the constant ξ as a function of a and L.

(b) Formulate the nonlocal eigenvalue problem that determines the stability of the solutions found in part (a). It should have the form

$$\lambda \phi = \phi'' - \phi + 2w\phi - \gamma \frac{\int_{-\infty}^{\infty} w\phi}{\int_{-\infty}^{\infty} w^2 dy} w^2$$

where γ is a constant that depends on a. What does Wei's theorem say about the stability of this eigenfunction?

(c) Consider the same eigenvalue problem as in part (b), but with boundary conditions for the eigenfunction of the form $\phi'(0) = 0$ and $\phi(L) = 0$. Both the steady state and the eigenvalue problem can be extended from $[0, L]$ to $[0, 2L]$, by reflecting at $x = L$. Then the steady state consists of two boundary spikes: half-spike at $x = 0$ and another half-spike at $x = 2L$. Show that there exists a critical length L_c such that this configuration is stable if $L > L_c$ and is unstable if $L < L_c$. Determine L_c. Hint: you should get the same eigenvalue problem as in (b), except γ will change.

(d) Use FlexPDE to compare the result you found in part (c) with full numerical simulations of the model. Illustrate your result by showing both stable and unstable configurations. See sample script on the website.

(e) Determine the asymptotic equation of motion of a single spike whose center is at x_0, inside the domain $[-L, L]$. Compare your result with full numerics using FlexPDE.