1. Consider the PDE

$$u_t = u_{yy} + 2u - 2u^3 \tag{1}$$

with the following boundary conditions:

$$u(-L,t) = -1, \quad u(+L,t) = +1.$$

Assume that L is large.

(a) This system admits an equilibrium solution in the form of an interface centered at the origin which, for large L, has the form $u(y,t) = u(y) \sim \tanh(y)$. The corresponding eigenvalue problem for the perturbation $u(y,t) = u(y) + e^{\lambda t}\phi(y)$ is

$$\lambda\phi = \phi_{yy} + 2\phi - 6\phi u^2$$

with *Dirichlet* boundary conditions, that is

$$\phi(\pm L) = 0.$$

Compute the largest eigenvalue λ explicitly. Is it negative or positive?

(b) Verify the stability of the problem by running the full numerics directly in FlexPDE. If you start with an interface off the center, does it move towards the boundary or towards the center? Are the dynamics consistent with what you obtained in part (a)? You can use the code from the course website.

(c) Consider the problem (1) but on a half-line with boundary conditions

$$u(0,t) = -1, \quad u(\infty,t) = 1.$$

For a solution of the form of an interface $u \sim \tanh(y - \xi(t))$ where $\xi \gg 1$, compute the asymptotic equations of motion of $\xi(t)$. Does ξ move towards the left boundary away from it? Compare your result with full numerics using FlexPDE.

2. In this problem you are asked to construct radially propagating interface in 2D to the PDE

$$u_t = \varepsilon^2 \Delta u - 2(u + \varepsilon a)(u-1)(u+1), \quad x \in \mathbb{R}^2.$$

where we assume that $0 < \varepsilon \ll 1$ and $a = O(1)$; we allow a to be either positive or negative.

(a) Find a travelling wave solution of the form

$$u(x) \sim U_0\left(\frac{r - r_0(\varepsilon^2 t)}{\varepsilon}\right), \ r = |x|,$$

where $U_0(z) = \pm\tanh(z)$. Derive an ODE for r_0; note that you get two different answers depending on whether you take $+\tanh$ or $-\tanh$. HINT: $\Delta u = u_{rr} + \frac{1}{r}u_r$.

(b) Under what conditions does the ODEs you derived in part (a) have an equilibrium point? Is that equilibrium stable or unstable?

3. Consider the Gray-Scott model,

$$u_t = \varepsilon^2 u_{xx} - u + Au^2 v$$
$$0 = v_{xx} - v + 1 - \frac{u^2 v}{\varepsilon}.$$

Here, $0 < \varepsilon \ll 1$ and $A = O(1)$.

(a) Construct a symmetric steady state on the domain $x \in [0, L]$ with Neumann boundary conditions for u, v, consisting of a half-spike centered at the origin. That is, to leading order, the solution for u should be of the form $u(x) \sim \xi w(x/\varepsilon)$ where $w(y) = \frac{3}{2}\operatorname{sech}^2(y/2)$ is the ground-state solution to $w_{yy} - w + w^2 = 0$. Your goal is to determine the value the constant ξ as a function of A and L. Show that ξ satisfies a quadratic equation. It has two solutions $\xi_- < \xi_+$ if $A > A_0$ or no solutions if $A < A_0$. Determine the value of A_0.

(b) Formulate the nonlocal eigenvalue problem that determines the stability of the solutions found in part (a). For $A > A_0$, show that one of the solutions you found in part (a) is unstable while the other is stable.

(c) Consider the same eigenvalue problem as in part (b), but with boundary conditions for the eigenfunction of the form $\phi'(0) = 0$ and $\phi(L) = 0$. Both the steady state and the eigenvalue problem can be extended from $[0, L]$ to $[0, 2L]$, by reflecting at $x = L$. Then the steady state consists of two boundary spikes: half-spike at $x = 0$ and another half-spike at $x = 2L$. Show that there exists a critical spike radius L_c such that this configuration is stable if $L > L_c$ and is unstable if $L < L_c$. Determine L_c.

(d) Use FlexPDE to compare the result you found in part (c) with full numerical simulations of the GS model. Illustrate your result by showing both stable and unstable configurations. See sample script on the website.

(e) Determine the asymptotic equation of motion of a single spike whose center is at x_0, inside the domain $[-L, L]$. Compare your result with full numerics using FlexPDE.