

Patch models in ecology

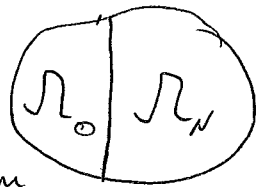
①

Q: Is organic agriculture sustainable?

- In conventional agriculture, pathogens [such as fungi, arthropods, bacteria...] are controlled using pesticides
- Pesticides are not used in organic farming.
- If organic farm is small and next to a conventional farm, pathogens may still be suppressed, due to effect of diffusion.
- However, if organic farm is too big, it may become a source of infestation.

Mathematical modelling:

- Assume the farm Ω is divided into two regions: $\bar{\Omega} = \bar{\Omega}_o \cup \bar{\Omega}_n$ representing organic and non-organic farm.
- Pathogen spreads through diffusion
- Pathogen grows logistically inside Ω_o
- Inside Ω_n , the growth of pathogen is suppressed using pesticides



Model:
$$\begin{cases} u_t = D \Delta u + g(x)u - u^2, & x \in \Omega \\ \partial_n u = 0, & x \in \partial\Omega \end{cases} \quad (1)$$

where $g(x) = \begin{cases} > 0, & x \in \Omega_0 \\ < 0, & x \in \Omega_N \end{cases}$ (growth rate) (2)

First, consider the zero-state $u \equiv 0$.

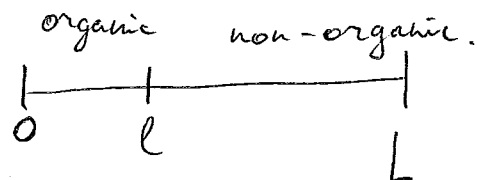
If we linearize around it, we get the problem

$$\begin{cases} \lambda \varphi = D \Delta \varphi + g(x) \varphi, & x \in \Omega \\ \partial_n \varphi = 0, & x \in \partial\Omega. \end{cases} \quad (3)$$

To get some insight, consider a one-dimensional case:

$$\bar{\Omega} = [0, L], \quad \bar{\Omega}_0 = [0, l], \quad \bar{\Omega}_N = [l, L]$$

Also let's take $D = 1$,



organic non-organic.

|-----|-----|

0 l L

$$g(x) = \begin{cases} +a, & x \in [0, l) \\ -b, & x \in (l, L] \end{cases}$$

Then (3) becomes

$$\begin{cases} \lambda \varphi = \varphi'' + a \varphi, & x \in (0, l) \\ \lambda \varphi = \varphi'' - b \varphi, & x \in (l, L) \\ \varphi'(0) = 0 = \varphi'(L) \\ \varphi(l^-) = \varphi(l^+), \quad \varphi'(l^-) = \varphi'(l^+) \end{cases}$$

On $(0, l)$: $\varphi = A \cos(\sqrt{a-\lambda} x)$, $\therefore$

On (l, L) : $\varphi = B \cosh(\sqrt{b+\lambda}(x-l))$

By scaling, take $A = 1$; continuity conditions:

$$\cos(\sqrt{a-\lambda} l) = B \cosh(\sqrt{b+\lambda}(l-l))$$

$$-\sqrt{a-\lambda} \sin(\sqrt{a-\lambda} l) = B \sqrt{b+\lambda} \sinh(\sqrt{b+\lambda}(l-l))$$

$$(*) \Rightarrow \boxed{-\sqrt{a-\lambda} \tan(\sqrt{a-\lambda} l) = \sqrt{b+\lambda} \tanh(\sqrt{b+\lambda}(l-l))}$$

Q: For given parameter values, is $\lambda < 0$?

Note that $\lambda < 0 \Rightarrow$ stability [no infestation]
 $\lambda > 0 \Rightarrow$ instability of zero state
 [outbreak occurs]

• Threshold happens if $\lambda = 0$

$$\Rightarrow \sqrt{a} \tan(\sqrt{a} l) = \sqrt{b} \tanh(\sqrt{b}(L-l))$$

$$\text{If } L \gg 1 \Rightarrow \sqrt{a} \tan \sqrt{a} l = \sqrt{b}$$

$$\Rightarrow \boxed{l_c = \frac{1}{\sqrt{a}} \arctan\left(\frac{\sqrt{b}}{\sqrt{a}}\right)}$$

• Can we show that $\lambda < 0$ if $l < l_c$
 $\lambda > 0$ if $l > l_c$?

- Use Rayleigh-Ritz formulation: the biggest eigenvalue of (3) satisfies:

$$(4) \quad \lambda = \max_{\varphi \in C^1(\Omega)} \frac{\int_{\Omega} -D|\nabla \varphi|^2 + g(x)\varphi^2}{\int_{\Omega} \varphi^2}$$

- Now Suppose that there is no organic plot, i.e. $g(x) < 0 \quad \forall x \in \Omega$. Then $\frac{\int_{\Omega} -D|\nabla \varphi|^2 + g(x)\varphi^2}{\int_{\Omega} \varphi^2} \leq 0 \quad \forall \varphi$
 $\Rightarrow \boxed{\lambda < 0}$
 $\Rightarrow$ no outbreak if no organic farm (as expected)

- On the other extreme, suppose the whole farm is organic, i.e. $g(x) > 0 \quad \forall x \in \Omega$. Then use $\varphi = 1$ as a test function in (4) to conclude that $\lambda \geq \text{avg}(g) > 0$

$\Rightarrow$ outbreak will occur if the average growth rate of pathogen is positive (in particular, if organic farm is sufficiently large).

- From (4) it also follows that increasing $g(x)$ will increase λ , i.e. increasing the size of organic farm will increase the chance of outbreak.

Non-zero steady states

Thm [Berestycki et al, 2004]:

Consider the model (1) and its steady state:

$$(5) \quad \begin{cases} 0 = D \Delta u + g(x)u - u^2, & x \in \Omega \\ \partial_n u = 0, & x \in \partial\Omega \end{cases}.$$

Let λ_0 be the biggest eigenvalue of the zero solution, i.e.

$$(6) \quad \begin{cases} \lambda_0 \phi_0 = D \Delta \phi_0 + g(x) \phi_0, & x \in \Omega \\ \partial_n \phi_0 = 0, & x \in \partial\Omega \end{cases}.$$

1. Suppose $\lambda_0 < 0$. Then the only sol'n to (5) is the zero solution $u=0$, and it is stable.
2. Suppose $\lambda_0 > 0$. Then there exists a unique positive sol'n to (5), $u(x) > 0 \forall x \in \Omega$.

Moreover, this solution is asymptotically stable, i.e. the corresponding eigenvalue prob,

$$(7) \quad \begin{cases} \lambda \phi = D \Delta \phi + g(x) \phi - 2u \phi, & x \in \Omega \\ \partial_n \phi = 0, & x \in \partial\Omega \end{cases}$$

admits only negative eigenvalues, $\lambda < 0$.

3. Suppose that $\Omega = B_R(0) = \{x : |x| < R\}$ and that $g(x) = g(|x|)$ and g is decreasing in $|x|$. Also suppose that $\lambda_0 > 0$ as in case 2. Then the positive solution is radially symmetric and decreasing, i.e. $u(x) = u(|x|)$ and $u'(|x|) \leq 0$.

(6)

Proof: To show 1., multiply (5) by φ_0

$$\int_{\Omega} D \varphi_0 (\Delta u + g u - u^2) = 0$$

Integrate by parts: $\int_{\Omega} \varphi_0 \Delta u = \int_{\Omega} \Delta \varphi_0 u + \underbrace{\int_{\partial \Omega} \partial_n u \varphi_0 - \partial_n \varphi_0 u}_{=0}$

$$\Rightarrow \int_{\Omega} u \underbrace{(D \Delta \varphi_0 + g \varphi_0)}_{\lambda \varphi_0} = \int_{\Omega} u^2 \varphi_0$$

$$(*) \quad \lambda \int_{\Omega} u \varphi_0 = \int_{\Omega} u^2 \varphi_0$$

Now if φ_0 is the eigenfunction corresponding to the biggest eigenvalue λ_0 , then $\varphi_0 \neq 0$

But then (*) is only possible if $\lambda_0 > 0$
or else $u \equiv 0$ $\blacksquare$

step 2:

Next to show existence of $u > 0$ in case $\lambda > 0$:

we use sub-super solutions. Let $\bar{u} = \max_{x \in \Omega} g(x)$.

$$\text{Then } D \Delta \bar{u} + g \bar{u} - \bar{u}^2 = (g - \bar{u}) \bar{u} \leq 0$$

So $\bar{u}$ is a supersolution of (5).

The subsolution is given by $\underline{u}(x) = \alpha \varphi(x)$

where α is to be chosen;

$$\begin{aligned} \text{Then } D\Delta \underline{u} + g(\underline{u} - \underline{u}^2) &= \alpha \lambda \varphi_0 - \alpha^2 \varphi_0^2 \\ &= \alpha \varphi_0 (\lambda - \alpha \varphi_0) \end{aligned} \quad (7)$$

Since $\lambda_0 > 0$ and WLOG, $\varphi_0 > 0$ inside Ω ,

$\underline{u}$ is subsolution provided that $\alpha \leq \frac{\lambda_0}{\max_{\Omega} \varphi_0}$.

Moreover we can make $\underline{u} \leq \bar{u}$ by

choosing $\alpha \leq \frac{\bar{u}}{\max_{\Omega} \varphi_0}$ so we choose:

$$\alpha = \min \left(\frac{\lambda_0}{\max_{\Omega} \varphi_0}, \frac{\bar{u}}{\max_{\Omega} \varphi_0} \right).$$

Then $\underline{u} \leq \bar{u}$ and the method of subsolutions / supersolutions [see for ex. Evans] guarantees the existence of a solution to (5)

with $\underline{u}(x) \leq u(x) \leq \bar{u}$. Also $u(x) > 0$ inside Ω since $\varphi_0(x)$ is.

Step 3: We prove stability of positive sol'n.
consider a sol'n $u(x) > 0$ of (5) and let (λ_0, φ_0) be the sol'n to (6) with $\lambda_0 > 0, \varphi_0 > 0$
[By step 1, such solution exists]

Now let (λ, φ) be the biggest eigenvalue sol'n to (7), so that $\varphi > 0$ inside Ω .

Multiply (7) by u and integrate:

$$\begin{aligned}\lambda \int_{\Omega} \varphi u &= \int_{\Omega} (\Delta \varphi u + g u \varphi - 2u^2 \varphi) \\ &= \int_{\Omega} (\Delta u + g u - 2u^2) \varphi \\ &= - \int_{\Omega} u^2 \varphi\end{aligned}$$

$$\Rightarrow \lambda = - \frac{\int_{\Omega} u^2 \varphi}{\int_{\Omega} \varphi u} < 0 \quad \text{since } \varphi > 0, u > 0 \text{ inside } \Omega.$$

Step 4: Uniqueness: The basic idea is that stability should somehow imply uniqueness.

(9)

Assume we have two solutions, $u_1, u_2 > 0$
 If $u_1 \neq u_2$ inside Ω then $\exists$ a point x s.t.
 WLOG $u_2(x) > u_1(x)$. Define

$$D = \{x : u_2(x) > u_1(x), x \in \Omega\}; D \neq \emptyset.$$

Let $w(x) = u_2(x) - u_1(x)$, so that

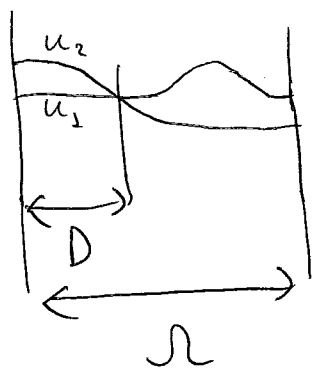
$$(9) \quad \Delta w + g w - (u_1 + u_2) w = 0$$

Now let φ_2, λ_2 be the principal eigenvalue /
 eigenvector pair sol'n to (7) with $u = u_2$;
 so that $\varphi_2 > 0$ inside Ω and $\lambda_2 < 0$ by
 step 3. Multiply (9) by φ_1 & integrate on D
 by parts:

$$0 = \int_{\partial D} (\varphi_1 \partial_n w - w \partial_n \varphi_1) + \int_D w (\underbrace{\Delta \varphi_1 + g \varphi_1}_{\lambda_1 \varphi_1 + 2u_1 \varphi_1}) - \int_D (u_1 + u_2) w \varphi_1$$

Now note that if $x \in \partial D$ then either $w(x) = 0$
 or else $x \in \partial \Omega$

In either case, $\partial_n w \leq 0$ and $w \partial_n \varphi_1 = 0$



so that the boundary term ≤ 0 .

$$\Rightarrow \int_D \lambda_1 \varphi_1 w + \int_D w \varphi_1 (\underbrace{2u_1 - (u_1 + u_2)}_{-w}) \geq 0$$

$$\Rightarrow \int_D \varphi_1 w (\lambda_1 - w) \geq 0, \text{ contradicting the fact that } \lambda_1 < 0, w > 0, \varphi_1 > 0.$$

(10)

Step 5: Radial symmetry: If $\Omega = B_R(0)$, $g(x) = g(|x|)$:

First, note that $u(x)$ must be radially symmetric; otherwise rotating it yields another solution which would violate uniqueness.

So $u(x) = u(r)$, $r = |x|$ satisfies:

$$\begin{cases} u_{rr} + \frac{n-1}{r} u_r + gu - u^2 = 0 & (10a) \\ u'(0) = 0, \quad u'(R) = 0 & (10b) \end{cases}$$

Then
(11) $u_{rrr} + \frac{n-1}{r} u_{rr} - \frac{n-1}{r^2} u_r + g'(r)u + (g - 2u)u_r = 0$

Now suppose that $u'(r) > 0$ for some $r \in (0, R)$.

Let r_0 be the point where $u'(r)$ has its max. Then $r_0 \in (0, R)$ because of (10b)

Thus $u'(r_0) > 0$; $u''(r_0) = 0$, $u'''(r_0) \leq 0$

So from (10a) we get $g(r_0) - u(r_0) \leq 0$;

from (11) we get: $g(r_0) - 2u(r_0) \geq 0$

$$\Rightarrow u(r_0) \leq 0$$

But this contradicts the assumption $u > 0$.



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[for a proof of sub-supersolution method]

Some Questions

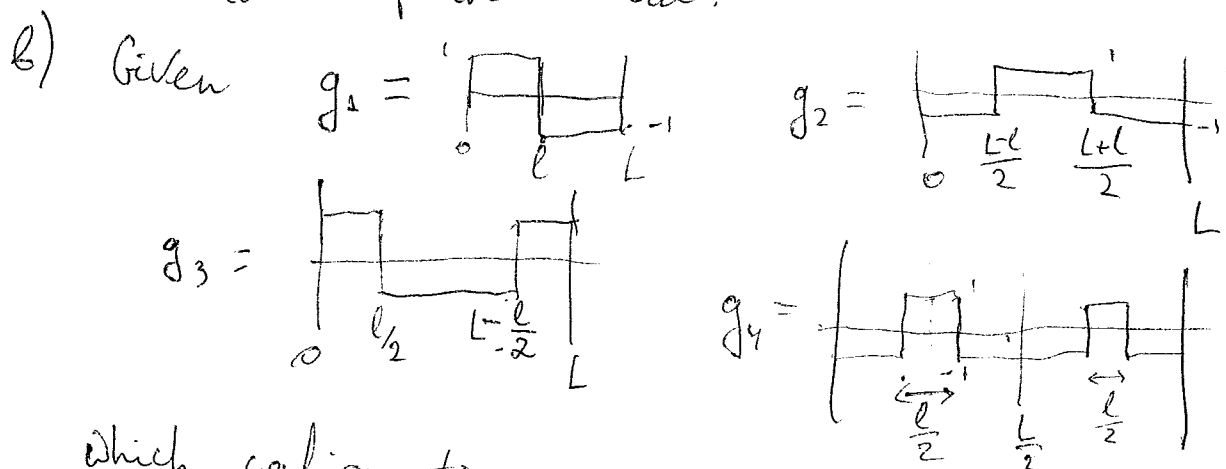
1) What is the optimal location for organic farm?

a) Consider the problem

$$\lambda \varphi = \varphi'' + g(x)\varphi, \quad \varphi'(0) = 0 = \varphi'(L) \quad (Q1)$$

$$\text{where } g(x) = \begin{cases} 1, & x \in (a, a+l) \\ -1, & x \in [0, L] \setminus (a, a+l) \end{cases}$$

Show that λ is minimized when $a = \frac{L-l}{2}$,
i.e. when the organic farm is located in the middle of the interval.



Which configuration minimizes λ ?

2) Write out the details for using sub-supersolution method to show existence of non-constant (outbreak) sol'n. of (5).

3) Consider equation (5) in the limit $D \gg 1$.

a) Show (formally) that

the outbreak occurs iff $\int g(x) > 0$ as $D \rightarrow \infty$.

b) Show that the outbreak sol'n is given by

$$u(x) \sim \bar{g} = \frac{\int_{\mathbb{R}} g}{\text{area}(\mathbb{R})} + O\left(\frac{1}{D}\right), \quad D \rightarrow \infty.$$

c) What can you say about (5) in the limit $D \ll 1$?

d) If we replace u^2 by $f(u)$ in (5), under what conditions on $f(u)$ is the theorem still valid?