MATH 5230/4230, Homework 1

Due: Wed, 29 Sept.

1. Let $u(x) = U(|x|)$, where $x \in \mathbb{R}^n$. Show that $\Delta u = w_{rr} + \frac{n-1}{r} w_r$ where $r = |x|$.

2. The *Bratu problem* is the following PDE problem that comes from combustion theory:

$$\begin{cases} \Delta u + \lambda e^u = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

where in general, Ω is any domain in $\mathbb{R}^n$. In the case where $\Omega = \{x : |x| < 1\}$ is a unit ball and u is radially symmetric, (1) reduces to

$$\begin{cases} u'' + \frac{n-1}{r} u' + \lambda e^u = 0, & 0 < r < 1 \\ u(1) = 0, \quad u'(0) = 0. \end{cases} \tag{2}$$

(a) Consider the radially symmetric solution in two dimensions. Show by an explicit computation that the solution to

$$\begin{cases} u'' + \frac{1}{r} u' + \lambda e^u = 0, & r > 0 \\ u(0) = a, \quad u'(0) = 0 \end{cases} \tag{3}$$

is given by

$$u(r) = a - 2\ln\left(1 + \frac{e^a}{8}\lambda r^2\right). \tag{4}$$

(b) Consider the boundary value problem

$$\begin{cases} u'' + \frac{1}{r} u' + \lambda e^u = 0, & r > 0 \\ u(1) = 0, \quad u'(0) = 0. \end{cases} \tag{5}$$

Using part (a), solve (5). Then sketch the bifurcation diagram by plotting λ on the x-axis and plotting $u(0)$ on the y-axis.

(c) From part (b), you should see that there is a critical value $\lambda = \lambda_c$ such that the solution does not exist for $\lambda > \lambda_c$ and there are two solutions for $\lambda < \lambda_c$. What is λ_c?

3. Now consider the $n-$dimensional problem (2) with $n = 1, 2$ or 3. The next few steps take you through the computations required to sketch the bifurcation diagram without having an explicit solution.

(a) First, make a transformation $u = \ln(v)$ in (2). What is the ODE that you obtain for v?

(b) Make a change of variables $v(r) = aV(R), \quad R = br$ for an appropriate choice of a, b so as to reformulate the boundary value problem (2) into an initial value problem.

(c) Solve the initial value problem you obtained in part (b) **numerically** and draw the resulting bifurcation diagram, plotting λ on the horizontal axis and $u(0)$ on the vertical axis, for dimensions $n = 1, 2, 3$. See the course website for a sample Maple worksheet on MEMS problem, which you can modify appropriately to do this. Double-check that the case $N = 2$ agrees with what you obtain in q.2.

(d) Make a change of variables of the form $R = e^t, \quad V(R) = e^{\alpha t} w(t)$ for an appropriate choice of α. Then define $p = \frac{w'}{w}; \quad q = w$. You get a first-order dynamical system for (p, q).

(e) Apply the phase plane analysis to the system you obtained in part (d) to obtain the desired bifurcation diagram in dimensions $n = 1, 2, 3$. Verify that your diagram is consistent with the numerical computations you obtained in part (c).

(f) In dimension $n = 3$, part (c) indicates that $\lambda \to \lambda^\star$ as $u(0) \to \infty$, for some $\lambda^\star > 0$. Using part (e), what is the value of $\lambda^\star$?