MATH 5230/4230, Homework 2

Due: Tues, 12 Oct.

1. With MEMS device modelling, we have shown in class the following sufficient conditions for existence of solution:

   Sufficient condition: Given a domain Ω, and a domain Ω' with $\Omega \subset \Omega'$, let μ, ϕ be the principal eigenvalue/eigenfucntion solution to

$$\begin{cases} \Delta\phi + \mu\phi = 0, & \text{inside } \Omega' \\ \phi = 0, & \text{on } \partial\Omega' \end{cases} \tag{1}$$

   with $\phi > 0$ inside Ω'. Suppose that

$$\frac{\min_{\Omega} \phi}{\max_{\Omega} \phi} \geq 0.4. \tag{2}$$

   Then there exists a solution to the MEMS problem provided that $\lambda < \mu/8$.

   (a) Suppose that $\Omega = (-L, L) \in \mathbb{R}$. Construct the domain Ω' which satisfies (2). What is is the corresponding μ?

   (b) Repeat question (a) but with $\Omega = B_L(0) \subset \mathbb{R}^2$, i.e. a disk of radius L. Hint: you can use Bessel functions for this.

   (c) Repeat question (b), but for an arbitrary domain $\Omega \subset \mathbb{R}^2$. Express the resulting μ as a function of $2L = \textit{diameter}\,(\Omega) = \max_{x,y\in\Omega} |x - y|$.

2. Consider the problem

$$\begin{cases} \Delta u + f(u) = 0, & \text{inside } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{3}$$

   where $f(u)$ is a smooth function satisfying $f(0) = 0$ and $f'(0) > \lambda$, where λ is the principal eigenvalue/eigenfucntion solution to

$$\begin{cases} \Delta\phi + \lambda\phi = 0, & \text{inside } \Omega \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

   Use the method of sub and supersolutions to show there exists a soltuion to (3).

3. Consider the eigenvalue problem for the farming problem,

$$\begin{cases} \lambda = D\Delta\phi + g(x)\phi, & \text{inside } \Omega \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{5}$$

   How does the principal eigenvalue λ change if $g(x)$ is increased? If D is increased? What are the consequences for the outbreak of diseases?

4. Let $g_1(x), \ldots g_4(x)$ be as shnown in Figure below. The region where g_i is positive corresponds to organic farm, the region where g_i is negative to conventional farm; note that the total area dedicated to organic farm and the total pesticide use is the same for all four choices g_i. For a fixed L and l, which of the four choices $g = g_i$ minimizes the principal eigenvalue λ of (5)? In other words, which

configuration minimizes the risk of the outbreak?