MATH 5230/4230, Homework 5

Due: Tues, Nov 16

1. Consider the Gray Scott model

$$v_t = v_{xx} - v + Av^2u$$
$$u_t = Du'' - u + 1 - v^2u.$$

The constant equilibria are $v_0 = 0$, $u_0 = 1$ and $v_\pm = \frac{A\pm\sqrt{A^2-4}}{2}$; $u_\pm = \frac{1}{Av_\pm}$. Show that the equilibrium v_+, u_+ is stable for all $A > 2$ iff $D < 2$ and is unstable if $D > 2$ and $A > A_c(D)$. Compute $A_c = A_c(D)$. Sketch the curve of $A_c(D)$.

2. The discrete aggregation model is

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{\substack{j=1\ldots N\\ j\neq i}} F(|x_j - x_i|)\frac{x_j - x_i}{|x_j - x_i|} \tag{1}$$

and the continuous version is

$$\rho_t + \nabla\cdot(v\rho) = 0;\quad v(x) = \int_{\mathbb{R}^n} F(|x-y|)\frac{x-y}{|x-y|}\rho(y)dy. \tag{2}$$

(a) For the discrete model (1), show that the center of mass is conserved. That is, let

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i;$$

then $\bar{x}(t) = \bar{x}(0)$ for all $t \geq 0$.

(b) For the continuous aggregation model in one dimension, (2), show that both the mass and the center of mass is conserved. That is, let

$$M = \int_{-\infty}^{\infty}\rho(y)dy;\quad N = \int_{-\infty}^{\infty} y\rho(y)dy;$$

then $M(t) = M(0)$ and $N(t) = N(0)$.

(c) Repeat part (b) for any dimension.

3. Consider the following one dimensional aggregation model:

$$\rho_t + (v\rho)_x = 0;\quad v(x) = -xa + \int_{-\infty}^{\infty}\exp(-|x-y|)\frac{x-y}{|x-y|}\rho(y)dy. \tag{3}$$

Here, a is a positive parameter. We seek to construct the steady states to the system (3)

(a) Let $G(x,y) = -e^{-|x-y|}$. Show that $G_{xx} - G = C\delta(x-y)$, where C is some constant. What is the constant C?

(b) We seek the steady states of (3) that is even i.e. $\rho(x) = \rho(-x)$ and that has has compact support, i.e. $\rho(x) = 0$ for $|x| > R$ and $v(x) = 0$ for $|x| < R$; where R is to be determined. Define

$$w(x) = -\frac{x^2}{2}a - \int_{-\infty}^{\infty}\exp(-|x-y|)\rho(y)dy.$$

Show that $w_{xx} - w = 2\rho - 2a + ax^2/2$, and that $w(x) = K$ for all $x \in [-R, R]$, for some constant K.

(c) Determine the constants K and R as well as the steady state $\rho(x)$ in terms of the mass $M = \int_{-R}^{R}\rho(x)dx$.