MATH 5230/4230, Homework 6

Due: Thurs, Nov 25

1. The Van der Pol oscillator is gouverned by the ODE

$$u_{tt} + \varepsilon u_t \left(u^2 - 1\right) + u = 0. \tag{1}$$

   (a) Using the method of multiple scales, show that the leading order expansion is of the form

   $$u = A(\varepsilon t) \cos\left(t + \Phi\left(\varepsilon t\right)\right)$$

   and find the equations for A and Φ.

   (b) Solve for A and Φ subject to initial conditions $u\left(0\right) = 1, \quad u_t\left(0\right) = 0.$

   (c) Use a computer to plot a numerical solution to (1) for $\varepsilon = 0.1$, $t \in \left(0, 100\right).$ On the same graph, plot the solution you found in part b and the envelope $A\left(\varepsilon t\right).$ Comment on what you observe. Note: If using maple, see sample code on the course webpage.

2. Perform multiple scales analysis on the ODE

$$u_{tt} + u = \varepsilon u'\left(u' - u\right); \quad u(0) = 1, \ u'(0) = 0.$$

   Sketch the solution and its envelope with $\varepsilon = 0.15$ and with $t = 0 \ldots 2000.$ What happens to the solution as $t \to \infty$?

3. A model of a laser subject to opto-electronic feedback is desribed by the following system [Erneux]:

$$x' = -y - \eta(1 + y(s - \theta)); \qquad y' = (1 + y)x. \tag{2}$$

   The parameters η and θ represent feedback parameter and delay time, respectively.

   (a) Linearize around the steady state $x = 0$, $y = -\eta/(1 + \eta)$. What is the transcendental equation for the resulting eigenvalue λ?

   (b) Seek Hopf bifurcations, i.e. plug in $\lambda = i\omega$, then separate real and imaginary parts. Show that Hopf bifurcations occur when

   $$\sin\left(\omega\theta\right) = 0 \quad \text{or} \quad (1 + \eta)\,\omega^2 = 1 + \eta\cos\left(\omega\theta\right). \tag{3}$$

   (c) Equations (3) represent curves in the (η, θ) plane. Plot these curves. Analytically classify any intersection points that you may observe. The first such point occurs when $\theta = 2\pi$, $\eta = 3/5.$ What are the possible values of ω at this point? List at least three more such intersections.

   (d) [BONUS] These intersection points are called double-hopf points and very interesting dynamics can be observed near these points. Perform numerical experiments of (2) with θ near 2π, and with η near $3/5$. Play around with η and θ near these values and describe what you observe. Hand in any useful plots of numerics.