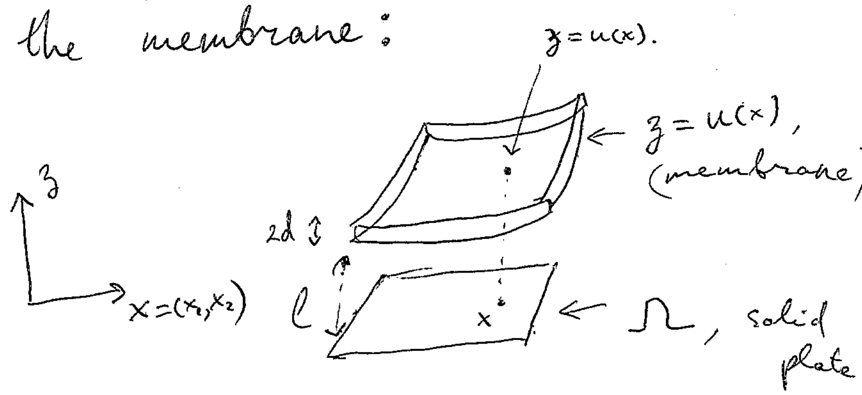
MEMS device modelling

- MEMS [Microelectromechanical system] device consists of an elastic membrane suspended above a rigid plate. A voltage is applied to the top of membrane; the potential difference causes the deflection of the membrane:



Assume:

$d \ll l \ll 1$;

Where
- $2d \equiv$ thickness of membrane
- $l \equiv$ dist. between plate and membrane
- $\Omega \equiv$ geometry of the rigid plate, with $z=0$;
- Membrane $\equiv \{ (x, z) : u(x) - d \le z \le u(x) + d \}$

Let $\Psi_i \equiv$ electrostatic potential inside membrane;

$\Psi_b \equiv$ " " between membrane & plate;

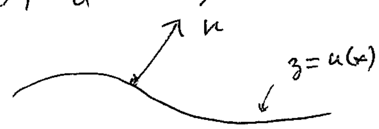
Model:

(1)
$$\begin{cases} \nabla \cdot (\nabla \Psi_b) = 0 \\ \Psi_b = 0 \text{ at } z = 0 \end{cases} \qquad \begin{cases} \nabla(\varepsilon_2 \nabla \Psi_i) = 0 \\ \Psi_i = V \text{ at } z = u(x) + d \end{cases}$$

(2)
$$\begin{cases} \Psi_b = \Psi_i \text{ at } z = u(x) - d \\ \varepsilon_0 \nabla \Psi_b \cdot \hat{n} = \varepsilon_2 \nabla \Psi_i \cdot \hat{n} \text{ at } z = u(x) - d \end{cases}$$

where $\hat{n} \equiv$ normal to $z = u(x) - d$

$\equiv c(-\nabla u, 1)$



$\varepsilon_0 \equiv$ permittivity of free space ; $\varepsilon_0 \equiv$ const.

$\varepsilon_2 \equiv$ " " membrane ; $\varepsilon_2(x, z) = \varepsilon_2(x)$;

Rescale : $\psi_{\substack{i\\b}} = \bar{\psi}_{\substack{i\\b}} V$; $z = \bar{z}\, l$

Drop the bars:

$$\Delta_x \psi_b + \frac{1}{l^2}\psi_{b_{zz}} = 0 \quad , \quad \nabla_x(\varepsilon_2 \nabla_x \psi_i) + \frac{\varepsilon_2}{l^2}\psi_{i_{zz}} = 0$$

$\psi_b = 0$ at $z = 0$, $\psi_i = 1$ at $z = u(x) + \frac{d}{l}$

$\psi_b = \psi_i$ at $z = u(x) - \frac{d}{l}$

$$-\nabla_x u \cdot \nabla_x\left(\psi_b - \frac{\varepsilon_2}{\varepsilon_0}\psi_i\right) + \frac{1}{l}\left(\psi_{b_z} - \frac{\varepsilon_2}{\varepsilon_0}\psi_{i_z}\right) = 0$$

Let $\delta = \frac{d}{l} \ll 1$; then at leading order we get:

$$(3) \quad \begin{cases} \psi_{b_{zz}} = 0 = \psi_{i_{zz}} \\ \psi_b = 0 \text{ at } z = 0, \quad \psi_i = 1 \text{ at } z = u + \delta \\ \frac{\varepsilon_2}{\varepsilon_0}\psi_{i_z} = \psi_{b_z} \text{ at } z = u - \delta \end{cases}$$

$$(4) \Rightarrow \begin{cases} \psi_b = A z \\ \psi_i = 1 + B\left(z - (u + \delta)\right) \end{cases}$$

and $A(u - \delta) = 1 + B(\underbrace{u - \delta - (u + \delta)}_{2\delta})$

$A = \frac{\varepsilon_2}{\varepsilon_0} B$

$$\Rightarrow \quad \Psi_i \sim 1 + \frac{z-u}{\frac{\varepsilon_2}{\varepsilon_0} u} \quad ; \quad \Psi_b \sim u$$

<u>Deflection equation:</u>

$$(5) \qquad T \Delta u = \frac{\varepsilon_2}{2} \left| \nabla \Psi_i \right|^2$$

$$\nabla \Psi_i = \left| \nabla_x \Psi_i \right|^2 + \underbrace{\left| \frac{V}{\ell} \Psi_{i_z} \right|^2}_{\frac{1}{\ell^2} \frac{\varepsilon_0^2}{\varepsilon_2^2} \frac{1}{u^2}} \qquad \text{[after rescaling]}$$

$\Rightarrow$

$$(6) \qquad \boxed{\begin{cases} \Delta u = \dfrac{\lambda f(x)}{u^2} , & x \in \Omega \\ u = 1 , & x \in \partial \Omega \end{cases}}$$

where $f(x) = \frac{\varepsilon_0^2}{\varepsilon_2^0}$; $\lambda = \frac{\varepsilon_0 V^2}{2 \ell^2 T}$

Note that $0 \leq f(x) \leq 1$.

- Experimentally, it is observed that the MEMS device becomes unstable if voltage is too high, and the membrane collapses. This is known as a <u>"pull-in"</u> voltage instability.
- Mathematically, solution to (6) ceases to exist if $\lambda > \lambda^*$ for some λ^*, the "pull-in" value.

Theorem 1: Suppose that $0 < c \leq f(x) \leq 1$.
Then there exists λ^* such that (6) has no solution if $\lambda > \lambda^*$. Moreover,

$$\lambda^* \geq \frac{4\alpha_1}{27c}$$ where α_1 is

the lowest eigenvalue of

$$(7) \quad \begin{cases} -\Delta\varphi = \alpha\varphi & \text{inside } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof: Rewrite (6) by shifting $u \to u-1$ as:

$$(8) \quad \begin{cases} \Delta u = \dfrac{\lambda f(x)}{(1+u)^2}, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

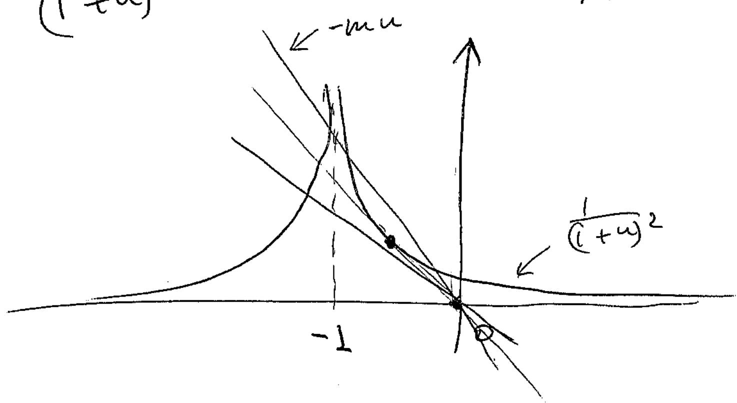
Let (φ_1, α_1) be principal eigenfunction/eigenvalue pair to (7). Then we can assume $\varphi_1 > 0$ inside Ω and $\alpha_1 > 0$. Multiply (8) by φ_1 & integrate by parts, we get:

$$(9) \quad \int_\Omega \varphi\left(\alpha_1 u + \frac{\lambda f(x)}{(1+u)^2}\right) = 0$$

Since $\varphi > 0$, the expression in brackets must change sign in order to satisfy (9).

Now $\quad (\;) \geq \alpha_1 u + \dfrac{\lambda c}{(1+u)^2}$

So consider $\quad m u + \dfrac{1}{(1+u)^2} = 0, \quad m = \dfrac{\alpha_1}{\lambda c}$. (10)



or $\quad -mu = \dfrac{1}{(1+u)^2}$. The double tangency occurs if $-m = \dfrac{-2}{(1+u)^3} \Rightarrow \dfrac{-(1+u)}{2} = u$

$$\Rightarrow u = -\frac{\frac{1}{2}}{1+\frac{1}{2}} = -\frac{1}{3}$$

$$\Rightarrow m = \frac{2}{\left(\frac{2}{3}\right)^3} = \frac{27}{4}$$

Thus (10) has no sol'n with $u > -1$ if $m < \dfrac{27}{4}$

or $\boxed{\lambda > \dfrac{27c}{4\alpha_1}}$

∎

Conclusion: the pull-in instability exists for any choice of the dielectric profile $f(x)$.

Next we show that sol'n to (6) exists if λ is sufficiently small, using sub-super solutions.

Def: A fcn $\bar{u}$ is <u>upper sol'n</u> if

$$(11)\quad \begin{cases} \Delta \bar{u} \le \dfrac{\lambda f(x)}{(1+\bar{u})^2} & \text{inside } \Omega \\ \bar{u} \ge 0 & \text{on } \partial\Omega . \end{cases}$$

It is <u>lower solution</u> if inequalities in (11) are reversed.

So any positive constant is an upper solution.

To find a <u>lower solution</u>, consider the principal eigfcn/value of:

$$(12)\quad \begin{cases} -\Delta \varphi = \mu \varphi & \text{inside } \Omega' \\ \varphi = 0 & \text{on } \partial\Omega' \end{cases}$$

Where $\Omega' \supset \Omega$ to be specified later; also take $\varphi > 0$ inside Ω' with $\max_{\Omega'} \varphi = 1$.

Our lower solution candidate is:

$$\underline{u} = A\varphi$$

where A is to be determined.

First, $A < 0$ since we want $\underline{u} \le 0$ on $\partial\Omega$.

Next,
$$\Delta \underline{u} - \frac{\lambda f}{(1+\underline{u})^2} = -A\mu\varphi - \frac{\lambda f}{(1+A\varphi)^2} \ge 0 \quad (13)$$

Since $\varphi \le 1$, we choose $(1 + A\varphi) > 0$

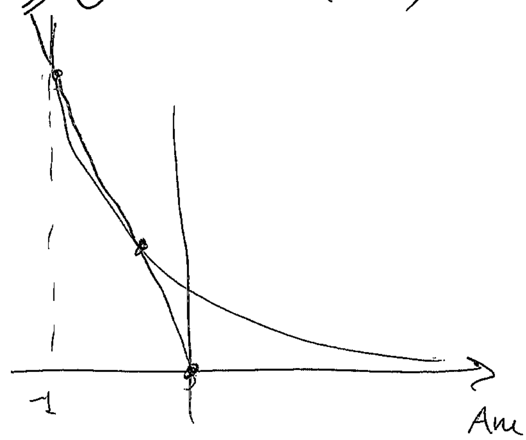
$\Rightarrow$ $-1 < A < 0$.

Finally, let $m = \min_{\Omega} \varphi$ so that

$m \leq \varphi \leq 1$ inside Ω. Then (13)

will be satisfied provided that

$$-\mu A m - \frac{\lambda}{(1+Am)^2} \geq 0 \qquad (14)$$



[recall that $f(x) < 1$]

As before, (14) can be satisfied

when $\dfrac{\mu}{\lambda} \geq \dfrac{27}{4}$; ~~in such a case,~~

and by taking $-Am \geq \frac{1}{3}$; since $-A \in (0,1)$

this yields a restriction $m \geq \frac{1}{3}$.

With this choice, $\underline{u}$ is a <u>lower solution</u>,

and $\underline{u} < 0 = \bar{u} \Rightarrow \exists$ sol'n to (6), $\underline{u} \leq u \leq 0$.

<u>We have proved:</u>

<u>Theorem 2:</u> Consider the principal sol'n to eigenvalue problem (12), where $\Omega \subset \Omega'$ and Ω' is chosen such that

$$\frac{\min_{\Omega} \varphi}{\max_{\Omega} \varphi} \geq \frac{1}{3}. \qquad (15)$$

Then (6) has a solution for all $\lambda \leq \frac{4}{27}\mu$.

Scaling analysis

Let's consider the radially symmetric case

$n=1$: $\Omega = (-L, L)$

$n=2$: $\Omega = B_1(0) = \{x : |x|^2 < L^2\}$

and $f(x) = 1$. Then (6) becomes:

$$(16) \quad \begin{cases} u_{rr} + \frac{n-1}{r} u_r = \frac{\lambda}{u^2}, & 0 < r < L \\ u'(0) = 0, \quad u(L) = 1. \end{cases}$$

We rescale:
$$\begin{cases} u(r) = a\, w(\eta) \\ \eta = b r \end{cases}$$

$$\Rightarrow \begin{cases} w_{\eta\eta} + \frac{n-1}{\eta} w_\eta = \frac{\lambda}{b^2 a^3} w^{-2} \\ w'(0) = 0, \quad w(bL) = 1/a \end{cases}$$

Now choose $\boxed{a = u(0)}$; $\frac{\lambda}{b^2 a^3} = 1$.

Then (16) becomes:

$$(17) \quad \begin{cases} (a) & w_{\eta\eta} + \frac{n-1}{\eta} w_\eta = w^{-2} \\ (b) & w'(0) = 0, \quad w(0) = 1 \end{cases}$$

along with:

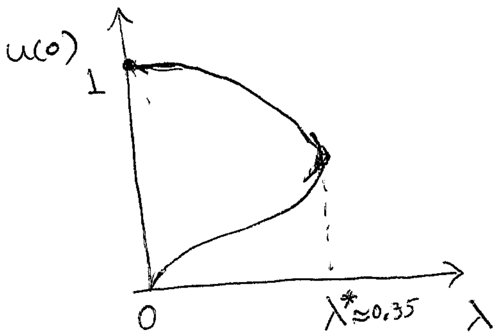
$$(18) \quad \begin{cases} a = u(0) = 1/w(bL) \\ \lambda = \frac{b^2}{[w(bL)]^3} \end{cases} ; \quad u(r) = w(br)$$

In other words, we have converted a boundary value problem (16) into an initial value problem (17) !

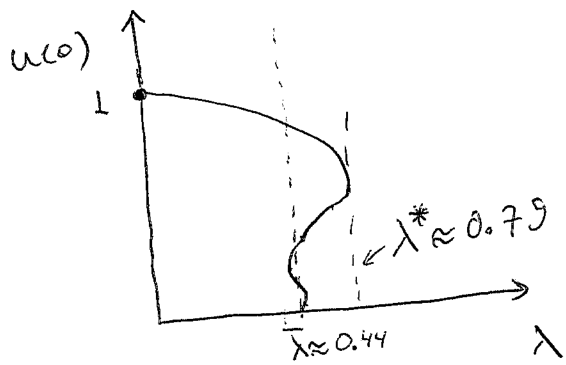
Numerically, (17) is much easier to solve than (16). This method is possible because of the underlying scaling symmetry of (16).

Using (17, 18), we can now draw a <u>bifurcation diagram</u> of (16), plotting $a = u(0)$ v.s. λ [parametrized by "b"]. Using Maple we get; with $L = 1$:

$n = 1$:



$n = 2$



This method gives λ^* numerically: $n = 1 : \lambda^* \approx 0.35$
$n = 2 : \lambda^* \approx 0.79$

- When $n = 1$, we observe that $\lambda \to 0$, $u(0) \to 0$ as $b \to \infty$
- When $n = 2$, we see that $\lambda \to \bar{\lambda} \approx 0.44$ as $b \to \infty$

  Moreover, the bifurcation curve seems to oscillate as $b \to \infty$, $\lambda \to \bar{\lambda}$. So <u>there exists infinitely many solutions</u> to (16) when $n = 2$, $\lambda = \bar{\lambda}$.

Can we show this?

• Note that (17a) has a scaling invariance
$\eta = \alpha \hat{\eta}$, $w = \beta \hat{w}$, whenever $\beta = \alpha^{2/3}$.

This means that a reduction of order is possible via a change of variables:

(19) $$\xi = \ln \eta \ , \quad w = \eta^{p} V$$

where p is to be specified. We compute:

$$\eta = e^{\xi}; \quad \begin{aligned} w &= e^{p\xi} V \\ w_\eta &= e^{-\xi + p\xi}(pV + V_\xi) \\ w_{\eta\eta} &= e^{(-2+p)\xi}\left(p(p-1)V + (2p-1)V_\xi + V_{\xi\xi}\right) \end{aligned}$$

$$\Rightarrow e^{(-2+p)\xi}\left\{V_{\xi\xi} + [(2p-1) + (n-1)]V_\xi + [p(p-1) + (n-1)p]V\right\}$$
$$= e^{-2p\xi} V^{-2}$$

$\Rightarrow$ choose $-2+p = -2p \Rightarrow \boxed{p = \frac{2}{3}}$

Then the indep. variable dissapears, and we get a 2-nd order dynamical system. To simplify further, let $P = \frac{1}{V}$, $q = \frac{V_\xi}{V}$; we get:

(20) $$\begin{cases} \dfrac{dP}{d\xi} = -Pq \\ \dfrac{dq}{d\xi} = -q^2 + P^3 - [n - \frac{2}{3}]q - [\frac{2}{3}(n - \frac{4}{3})] \end{cases}$$

In terms of η, w we have:

(21) $$\eta = e^{\xi} \ ; \quad P = \frac{\eta^{2/3}}{w} \ ; \quad q = \eta \frac{w_\eta}{w} - \frac{2}{3}$$

So initial conditions (17 b) become:

(22) $$\xi \to -\infty; \quad p \to 0, \quad q \to -\frac{2}{3}.$$

Now consider the phase portrait of (20):

- Steady states are:

$$A: \; p = 0, \quad q = -\frac{2}{3}$$

$$B: \; p = 0, \quad q = \frac{4-3n}{3}$$

$$C: \; p = \left[\frac{2}{3}\left(n - \frac{4}{3}\right)\right]^{\frac{1}{3}}, \quad q = 0$$

Note that A corresponds precisely to (22)!

Linearize (20):

$$J = \left(\begin{array}{c|c} -q & -p \\ \hline 3p^2 & -2q - \left(n - \frac{2}{3}\right) \end{array}\right)$$

$$J|_A = \begin{pmatrix} 2/3 & 0 \\ 0 & 2-n \end{pmatrix} \qquad J|_B = \begin{pmatrix} \frac{-4+3n}{3} & 0 \\ 0 & \frac{-3n+2}{3} \end{pmatrix}$$

$$J|_C = \begin{pmatrix} 0 & -p_0 \\ 3p_0^2 & -n + \frac{2}{3} \end{pmatrix}, \quad \text{where } p_0 = \left(\frac{2}{3}\left(-\frac{4}{3} + n\right)\right)^{\frac{1}{3}}.$$

Treat n as a bifurcation parameter; $n \geq 1$.

Stability: A: $\lambda = \frac{2}{3}, \; 2-n$

- Source if $n < 2$
- Saddle if $n > 2$

B:

- Sync if $n < \frac{4}{3}$
- Saddle if $n > \frac{4}{3}$

C: $\det J_c = 3p_0^3 = 2(-\frac{4}{3}+n)$ $\begin{cases} <0, & n<\frac{4}{3} \\ >0, & n>\frac{4}{3} \end{cases}$

$\operatorname{tr} J_c = \frac{2}{3} - n < 0$

$\operatorname{tr}^2 - 4\det = n^2 - \frac{28}{3}n + \frac{100}{9}$ $\begin{cases} >0, & n=1 \\ <0, & n=2,\dots,7 \\ >0, & n \geq 8 \end{cases}$

So C is
- Saddle if $n=1$
- Spiral sync if $n = 2, 3 \dots 7$
- Sync if $n \geq 8$

See phase plots for $n=1, 2$ on next page.

Now we want to know what happens to (18) as $b \to \infty$, i.e. behaviour of $\omega(\eta)$ as $\eta \to \infty \Rightarrow \xi \to \infty$.

Moreover, $\omega > 0$ so $p > 0$. From phase plane, any sol'n that starts with A as $\xi \to -\infty$, must end up at B as $\xi \to +\infty$ (if $n=1$) or at C as $\xi \to +\infty$ (if $n \geq 2$)

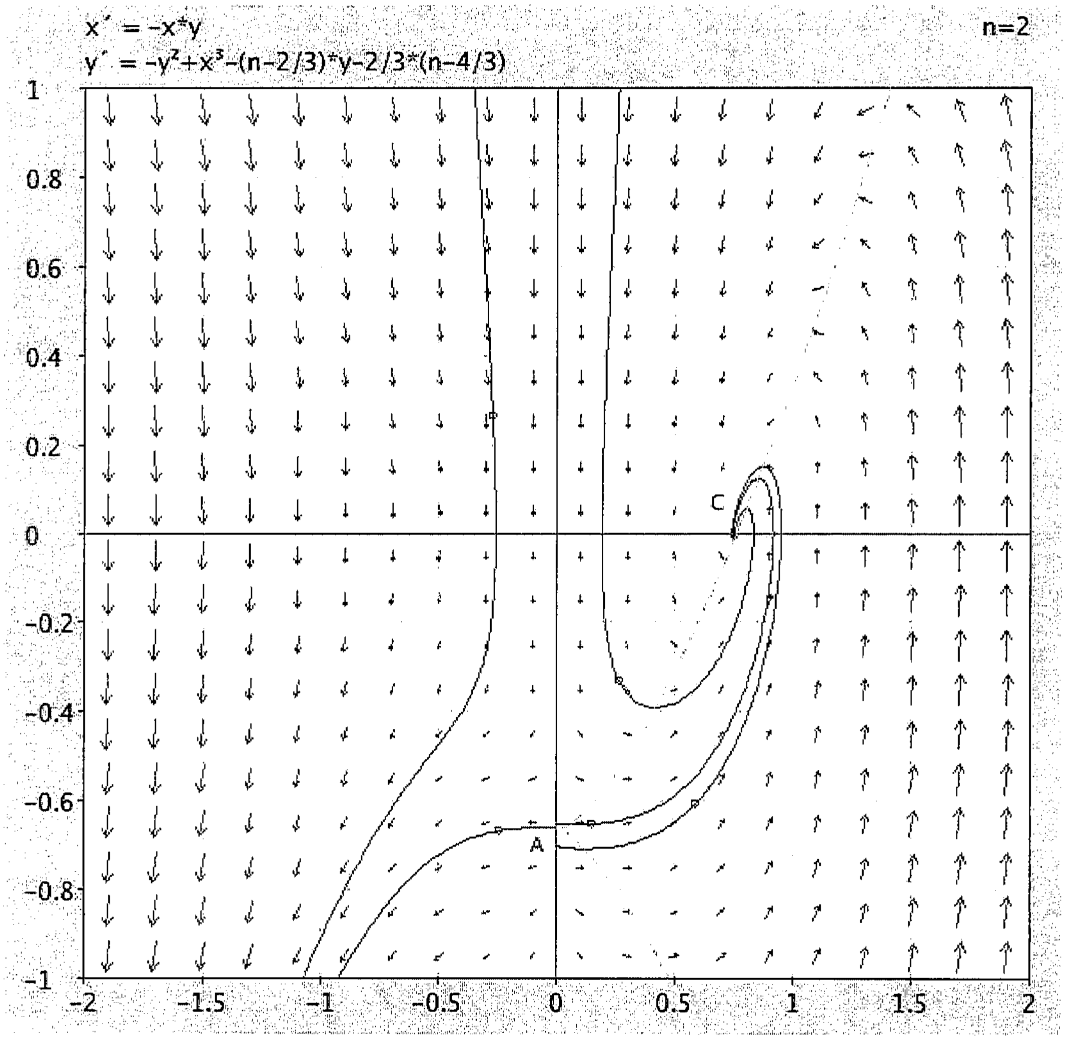
If $n=1$, near B we get

$$\begin{pmatrix} p \\ q \end{pmatrix} \sim C_1 e^{-\frac{1}{3}\xi} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-\frac{4}{3}\xi} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

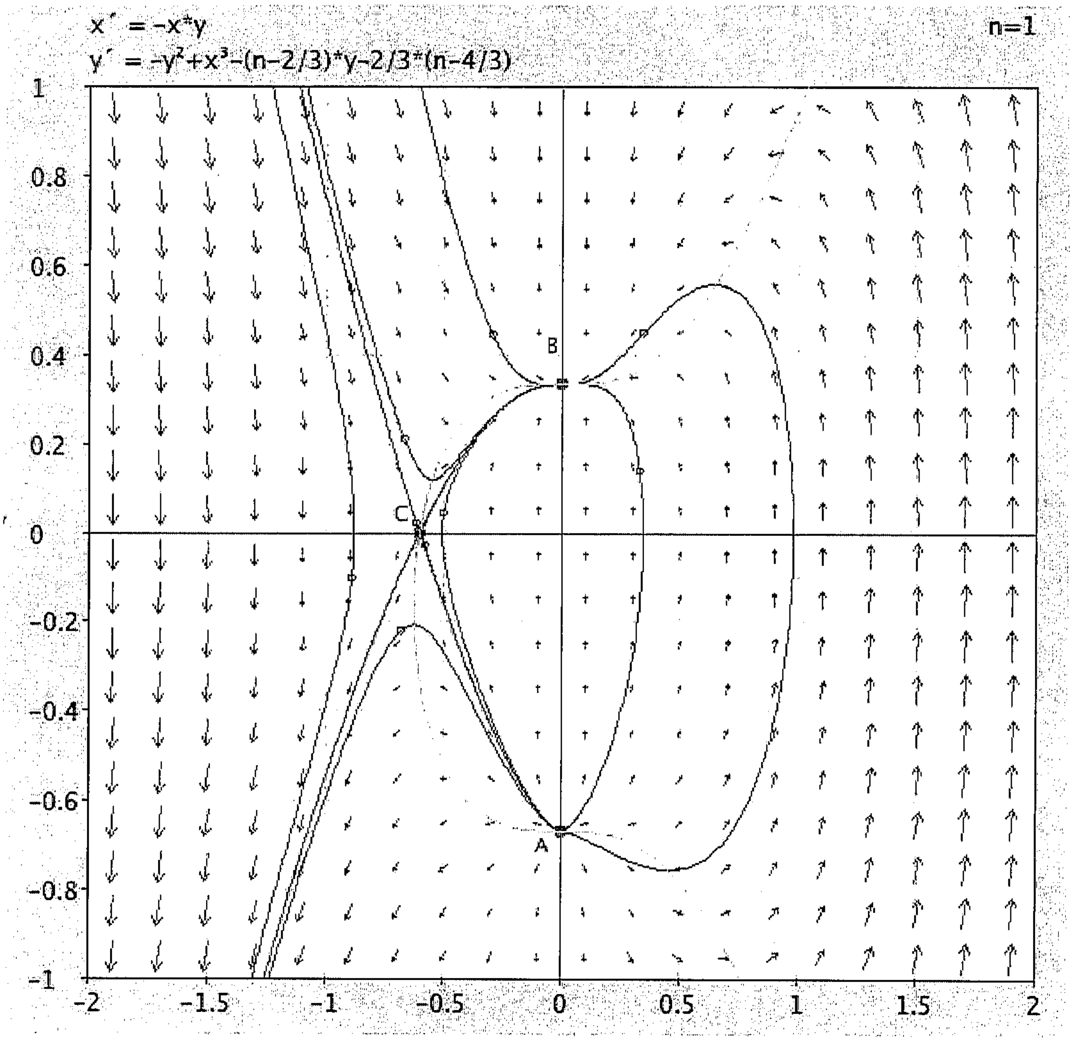
$\Rightarrow \beta\omega \sim C_0 \eta$ as $\eta \to \infty$.

$\Rightarrow \begin{cases} a \sim \frac{1}{C_0 \beta L}, \quad \lambda \sim \frac{1}{C_0^3 b L^2} \end{cases}$ as $b \to 0$

So $\lambda \to 0$, $u(0) \to 0$ (see figure with $n=1$ on p. 9)



x´ = −x*y
y´ = −y²+x³−(n−2/3)*y−2/3*(n−4/3)
n=2
A
C
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2



x´ = −x*y
y´ = −y²+x³−(n−2/3)*y−2/3*(n−4/3)
n=1
A
B
C
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2

<u>If $n=2$</u>, near C we get:

$$\lambda = -\frac{2}{3} \pm i\, 1.4337$$

$$\Rightarrow \quad p \sim e^{-\frac{2}{3}\xi}\left[A\cos(1.43\,\xi) + B\sin(1.43\,\xi)\right] + \left(\frac{4}{9}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad w(\eta) \sim \eta^{\frac{2}{3}}\left(\frac{9}{4}\right)^{\frac{1}{3}} + \text{some small oscillations}$$

$$\Rightarrow \begin{cases} \lambda \sim \dfrac{1}{L^2}\,\dfrac{4}{9} + \text{some oscillations as } b \to \infty \\ u(0) \sim c\left(b\right)^{-\frac{2}{3}} \to 0 \quad \text{as } b \to 0 \end{cases}$$

This shows that $\bar{\lambda} = \frac{4}{9} \approx 0.44$ in figure $n=2$) of page 9.

Some questions

1) Consider condition (15) of theorem 2. Can (15) be satisfied?

a) If $\bar{\Omega} = [-\ell, \ell]$, show that

$$\mu = \frac{\alpha}{\theta_0^2} \quad \text{where} \quad \cos\theta_0 = \frac{1}{3}$$

and α as in theorem 1. Conclude that if $f(x) = 1$ then $\lambda^* \in \frac{4}{27} \frac{\pi^2}{\ell^2} \left[\frac{1}{\theta_0^2}, 1\right]$.

b) If $\Omega \subset \mathbb{R}^2$, how can you choose Ω' to satisfy (15)?

2) Consider the problem:

$$(*) \quad \begin{cases} u_{rr} + \frac{n-1}{r} u_r + \lambda e^u = 0 \\ u'(0) = 0, \quad u(1) = 0, \quad u > 0 \text{ in } [0,1). \end{cases}$$

Sketch the bifurcation diagram $u(0)$ vs. λ for $n = 1, 2, 3$. Show that $(*)$ has infinitely many solutions if $n = 3$ and $\lambda = 2$, but at most two solutions for any λ if $n \leq 2$.

References

1) J. A. Pelesko, Mathematical modelling of electrostatic MEMS with tailored dielectric properties,
SIAM J. Appl. Math, Vol 62 No 3 pp. 888-908

2) D.D. Joseph & T.S. Lundgren,
Quasilinear Dirichlet Problems driven by positive sources,
Arch. Rat. Mech. Anal., Vol. 49