# Method of super-sub-solutions; maximum principle

## Max principle:

Suppose that
$$\begin{cases} \Delta w - \lambda w \leq 0 & \text{inside } \Omega \\ w \geq 0 & \text{on } \partial\Omega \end{cases} \qquad (*)$$

where $\lambda > 0$. Then $w \geq 0$ for all $x \in \Omega$.

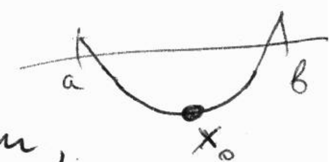
Proof: Let us first demonstrate this for 1-D; $\Omega \in \mathbb{R}^1$
we will then generalize to any dimension.

For 1-D, the problem $(*)$ becomes:
$$\begin{cases} w'' - \lambda w \leq 0 & \text{for } x \in (a, b) \\ w \geq 0 & \text{for } x = a \text{ or } x = b \end{cases} \qquad (**)$$

Now let $x_0 \in [a, b]$ be the minimizer of $w$,

i.e. $\min_{[a,b]} w = w(x_0)$.

- If $x_0 = a$ or $b$ then we are done since $w \geq 0$ at $x=a$
- If $x_0 \in (a, b)$, suppose $w(x_0) \leq 0$.




But since $x_0$ is the minimum,
we have $w''(x_0) \geq 0 \Rightarrow w''(x_0) - \lambda w > 0$
which contradicts $(**)$

$\Rightarrow w(x_0) > 0$

In higher dimensions, note that if $x_0$ is an interior min of $u(x)$, then the Hessian matrix

$$H(u) = \begin{bmatrix} \partial_{x_1x_1} u & \partial_{x_1x_2} u & \cdots & \partial_{x_1x_n} u \\ \vdots & & & \\ \partial_{x_nx_1} u & & \cdots & \partial_{x_nx_n} u \end{bmatrix}$$

is definite positive at $x = x_0$

[i.e. all eigenvalues of $H(u)$ are positive]

In particular, $\Delta u = u_{x_1x_1} + \cdots + u_{x_nx_n}$

$$= \text{trace } H(u) \geq 0$$

since trace $H$ = sum of eigenvalues of $H$.

The rest of the proof is exactly the same as in 1-D ∎

Remark: Max principle actually holds even if $\lambda = 0$, but this is harder to prove [ $\lambda = 0$ case is called "strong max principle"].

# Method of sub-super solutions:

Consider: $$\begin{cases} \Delta u - \lambda u = g(u) & \text{inside } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (*)$$

Def: $\bar{u}$ is upper sol'n of (1) if $\begin{cases} \Delta \bar{u} - \lambda \bar{u} \le g(\bar{u}) & \text{in} \\ \bar{u} \ge 0 & \text{on } \partial\Omega \end{cases}$

$\underline{u}$ is lower sol'n of (1) if $\begin{cases} \Delta \underline{u} - \lambda \underline{u} \ge g(\underline{u}) & \text{in} \\ \bar{u} \le 0 & \text{on } \partial\Omega. \end{cases}$

Claim: Suppose $\lambda \ge 0$, $g(u)$ is decreasing and $\bar{u}$, $\underline{u}$ are upper and lower sol'n of (*)

Then (*) has a unique solution $u$ ~~bett~~ between $\underline{u}(x) \le u(x) \le \bar{u}(x)$.

Pf: Let $u_0 = \underline{u}$ and define $u_1$ to solve:

$$\begin{cases} \Delta u_1 - \lambda u_1 = g(u_0) & \text{inside } \Omega \\ u_1 = 0 & \text{on } \partial\Omega \end{cases}$$

similarly define $\begin{cases} \Delta u_i - \lambda u_i = g(u_{i-1}) & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$

Step 1: $\underline{u} = u_0 \leq u_1 \leq u_2 \cdots$

Pf: Let $w = u_1 - u_0$; then

$$\begin{cases} \Delta u_1 - \lambda u_1 = g(u_0) \\ \Delta u_0 - \lambda u_0 \geq g(u_0) \end{cases}$$

$$\Rightarrow \begin{cases} \Delta w - \lambda w \leq 0 & \text{inside } \Omega \\ w \geq 0 & \text{on } \partial\Omega \end{cases}$$

$\Rightarrow$ $w \geq 0$ in $\bar{\Omega}$ [by max. principle]

$\Rightarrow$ $u_1 \geq u_0$

Similarly: $\Delta u_2 - \lambda u_2 = g(u_1)$

$\Delta u_1 - \lambda u_1 = g(u_0) \geq g(u_1)$

[because $g(u)$ is decr. and $u_0 \leq u_1$]

$\Rightarrow$ $u_2 \geq u_1$ [as before]

Step 2: $u_i \leq \bar{u}$. pf: Let $w = \bar{u} - u_i$; then

$$\begin{cases} \Delta \bar{u} - \lambda \bar{u} \leq g(\bar{u}) \\ \Delta u_i - \lambda u_i = g(u_{i-1}) \geq g(\underline{u}) \end{cases}$$ [since $g$ is decr.]

$\Rightarrow$ $\Delta w - \lambda w \leq g(\bar{u}) - g(\underline{u}) \leq 0$ inside $\Omega$

and $w \geq 0$ on $\partial\Omega$ $\Rightarrow$ $\boxed{w > 0}$ by max princip

<u>Step 3:</u> We have $u_i(x) \to u(x)$ as $i \to \infty$ ~~since~~ pointwise [since the seq. $\{u_i(x)\}_{i=0}^{\infty}$ is bounded and monotone].

It can be also shown that $u(x) \in C^2(\Omega)$ [outside the scope of this course].

This proves <u>existence</u>. $\blacksquare$