Turing instability

Q: Can diffusion cause an instability?

- For the heat eq'n, $u_t = D\Delta u$, we know that any initial conditions decay to a constant steady state, so diffusion has a smoothing effect.
- More generally, consider a pde

$$(1) \qquad \begin{cases} u_t = D u_{xx} + f(u), & x \in (0,L) \\ u'(0) = 0 = u'(L) \end{cases}$$

Suppose u_0 is a stable steady state of ODE $u_t = f(u)$; i.e.

$$(2) \qquad f(u_0) = 0; \quad f'(u_0) < 0$$

Can addition of diffusion $D u_{xx}$ destabilize u_0?

Linearize: $u(x,t) = u_0 + e^{\lambda t}\varphi(x)$; $\varphi \ll 1$;

$$\begin{cases} \lambda \varphi = D\varphi_{xx} + f'(u_0)\varphi \\ \varphi'(0) = 0, \ \varphi'(L) = 0 \end{cases}$$

$\Rightarrow \varphi = \cos(mx)$ where $mL = k\pi$, $k = 1, 2, 3 \ldots$

and $\lambda = -m^2 D + f'(u_0) < 0$ from (2)

Conclusion: u_0 remains stable;

For a <u>single PDE</u> (2), diffusion cannot cause instability.

In 1952 paper, Turing asked: can diffusion destabilize a <u>system</u> of PDE's:

$$(3) \quad \begin{cases} u_t = D_1 u_{xx} + f(u,v) \\ v_t = D_2 v_{xx} + g(u,v) \end{cases}$$

Suppose (3) has a homogeneous steady state (u_0, v_0) satisfying $f(u_0, v_0) = 0$, $g(u_0, v_0) = 0$. Moreover, suppose (u_0, v_0) is <u>stable</u> s.s. of the corresponding ODE system:

$$(4) \quad \begin{cases} u_t = f(u,v) \\ v_t = g(u,v) \end{cases}$$

Linearize (4): $u(t) = u_0 + e^{\lambda t}\eta$; $v(t) = v_0 + e^{\lambda t}\xi$

$$(5) \quad \Rightarrow \quad \lambda \begin{bmatrix} \eta \\ \xi \end{bmatrix} = \underbrace{\begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}\Bigg|_{(u,v)=(u_0,v_0)}}_{J} \begin{bmatrix} \eta \\ \xi \end{bmatrix}$$

So (4) is <u>stable</u> provided that all eigenvalues of J have negative real part, or

$$\det J > 0; \quad \operatorname{tr} J < 0.$$

$$\Longleftrightarrow \quad \begin{cases} f_u g_v - f_v g_u > 0 \\ f_u + g_v < 0 \end{cases} \qquad (6)$$

(where the expressions are evaluated at $u = u_0$, $v = v_0$).

Now linearize (3):

$$(7) \quad \begin{cases} u = u_0 + \cos(mx)\, e^{\lambda t} \eta \\ v = v_0 + \cos(mx)\, e^{\lambda t} \xi \end{cases} \qquad \text{with } \eta, \xi \ll 1$$

$$(8) \quad \Rightarrow \quad \lambda \begin{bmatrix} \eta \\ \xi \end{bmatrix} = \underbrace{\left(\begin{bmatrix} -m^2 D_1 & 0 \\ 0 & -m^2 D_2 \end{bmatrix} + \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} \right)}_{M} \begin{pmatrix} \eta \\ \xi \end{pmatrix}$$

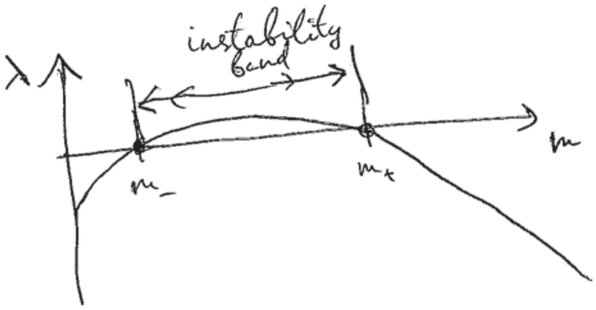
Now λ satisfies: $\lambda^2 - (\mathrm{tr}\, M)\lambda + \det M = 0$.

- If $m = 0 \Rightarrow \mathrm{Re}(\lambda) < 0$ [since ODE is stable]
- If $m \to \infty \Rightarrow M \sim -m^2 \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} \Rightarrow \mathrm{Re}\,\lambda < 0$

Is there a range of m for which $\mathrm{Re}\,\lambda > 0$?

Suppose that such a range exists. Its endpoints satisfy:
- either $\det M = 0$ [one of $\lambda = 0$]
- or $\mathrm{tr}\, M = 0$ [$\lambda = \pm i\omega$, $\mathrm{Re}\,\lambda_\pm = 0$]

<u>Exercise</u>: only $\det M = 0$ is possible.



Then $m_\pm$ satisfy:

$$\left(f_u - m^2 D_1\right)\left(g_v - m^2 D_2\right) - f_v g_u = 0$$

$$\Leftrightarrow \quad m^4 D_1 D_2 + m^2\left(-D_1 g_v - D_2 f_u\right) + f_u g_v - f_v g_u = 0$$

This is a quadratic in m^2; thus instability band exists iff

(9) $D_1 g_v + D_2 f_u > 0$ and $\left(D_1 g_v + D_2 f_u\right)^2 - 4 D_1 D_2 \left(f_u g_v - f_v g_u\right) \geq 0$

Let $r = \dfrac{D_2}{D_1}$. Then (9) becomes:

(10) $p(r) = r^2 f_u^2 + r\left(4 f_v g_u - 2 f_u g_v\right) + g_v^2 \geq 0$

and $\quad g_v + r f_u > 0$

Note that $p(r) > 0$ if $r \gg 1$ or $r = 0$

Thus instability band will appear if

- $f_u > 0$ and $\dfrac{D_2}{D_1}$ is sufficiently large
- $g_v > 0$ and $\dfrac{D_1}{D_2}$ is sufficiently large.

Exercise: No istability if $f_u < 0$, $g_v < 0$ or if $D_1 = D_2$.

<u>Example</u> Consider <u>GS model</u> :

$$(11)\quad \begin{aligned} v_t &= D_1 v'' - v + A v^2 u \\ \tau u_t &= D_2 u'' - u + 1 - v^2 u \end{aligned}$$

S.S. given by: $v_0 u_0 A = 1$, $v_0^2 - A v_0 + 1 = 0$

<u>or</u> $v_0 = 0$, $u_0 = 1$

$$\Rightarrow \begin{cases} v_0 = 0, \; u_0 = 1 \\ v_0^{\pm} = \dfrac{A \pm \sqrt{A^2 - 4}}{2}, \quad u_0^{\pm} = \dfrac{1}{A v_0^{\pm}}, \quad A > 2 \end{cases}$$

<u>Linearize</u>: $v = v_0 + \cos(mx) e^{\lambda t} \xi$
$u = u_0 + \cos(mx) e^{\lambda t} \eta$

$\Rightarrow$ λ is an eigenvalue of

$$M = \begin{bmatrix} -D_1 m^2 - 1 + 2u_0 v_0 & v_0^2 A \\ -\frac{1}{\tau} 2 u_0 v_0 & -\frac{1}{\tau}\left(D_2 m^2 + 1 + v_0^2\right) \end{bmatrix}$$

For s.s. $v_0 = 0, u_0 = 1$, M is diagonal with $\lambda < 0$ $\forall m$.

For s.s. $v_0^{\pm}, u_0^{\pm}$ we have

$$M = \begin{bmatrix} -D_1 m^2 + 1 & v_0^2 A \\ -\frac{1}{\tau} \frac{2}{A} & -\frac{1}{\tau}\left(D_2 m^2 + 1 + v_0^2\right) \end{bmatrix}$$

Let $f(m) = \tau \det M = D_1 D_2 m^4 + m^2\left(D_1(1+V_0^2) - D_2\right) + V_0^2 - 1$

so that $f(0) = V_0^2 - 1 \begin{cases} < 0 & \text{if } V_0 = V_0^- \\ > 0 & \text{if } V_0 = V_0^+ \end{cases}$

So V_0^- is unstable even when $m = 0$;

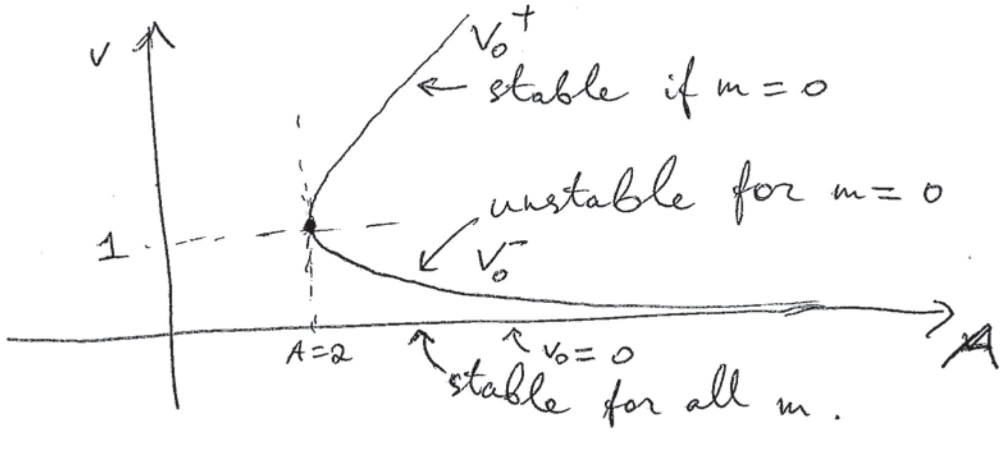
Suppose τ is s.t. $\text{trace } M\Big|_{\substack{m=0 \\ V_0 = V_0^+}} < 0$

i.e. $\tau < A V_0^+$

In particular, take $\tau < 2$.

Then V_0^+ is stable w.r.t. $m = 0$

So for $m=0$ we have the bifurcation diagram:



Q: Can the V_0^+ become unstable for some m?

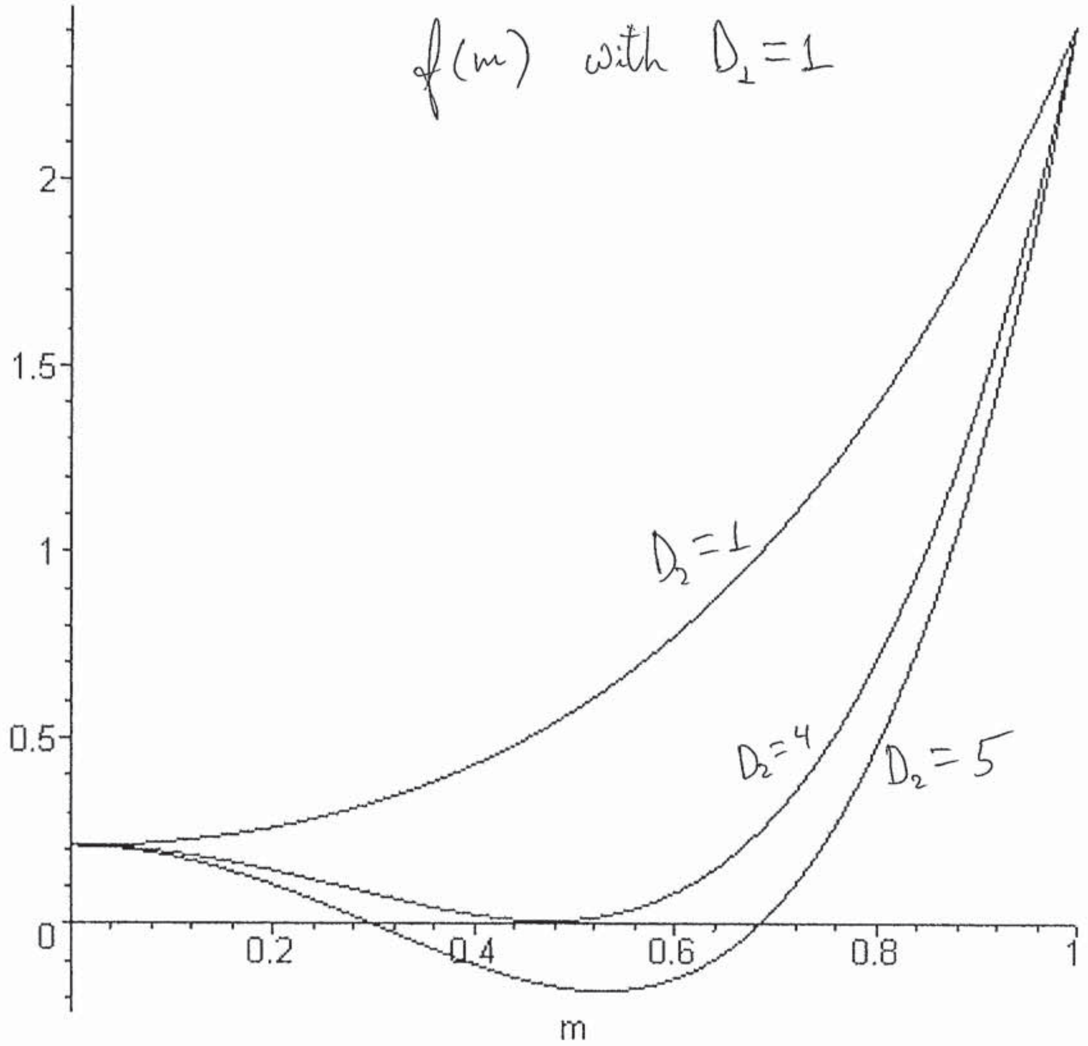
Answer: By assumption $\tau < 2$, $\operatorname{tr} M < 0 \ \forall m$.

So mode m is unstable iff $f(m) < 0$

An instability band appears as D_2 is increased

For example, take $D_1 = 1$ and $V_0 = 1.1$.

Then $f(m)$ is as shown for $D_2 = 1, 4, 5$:



An instability band appears around $D_2 \sim 4$.

For example, when $D_2=5$, the modes $m \in [0.3, 0.68]$ are unstable.

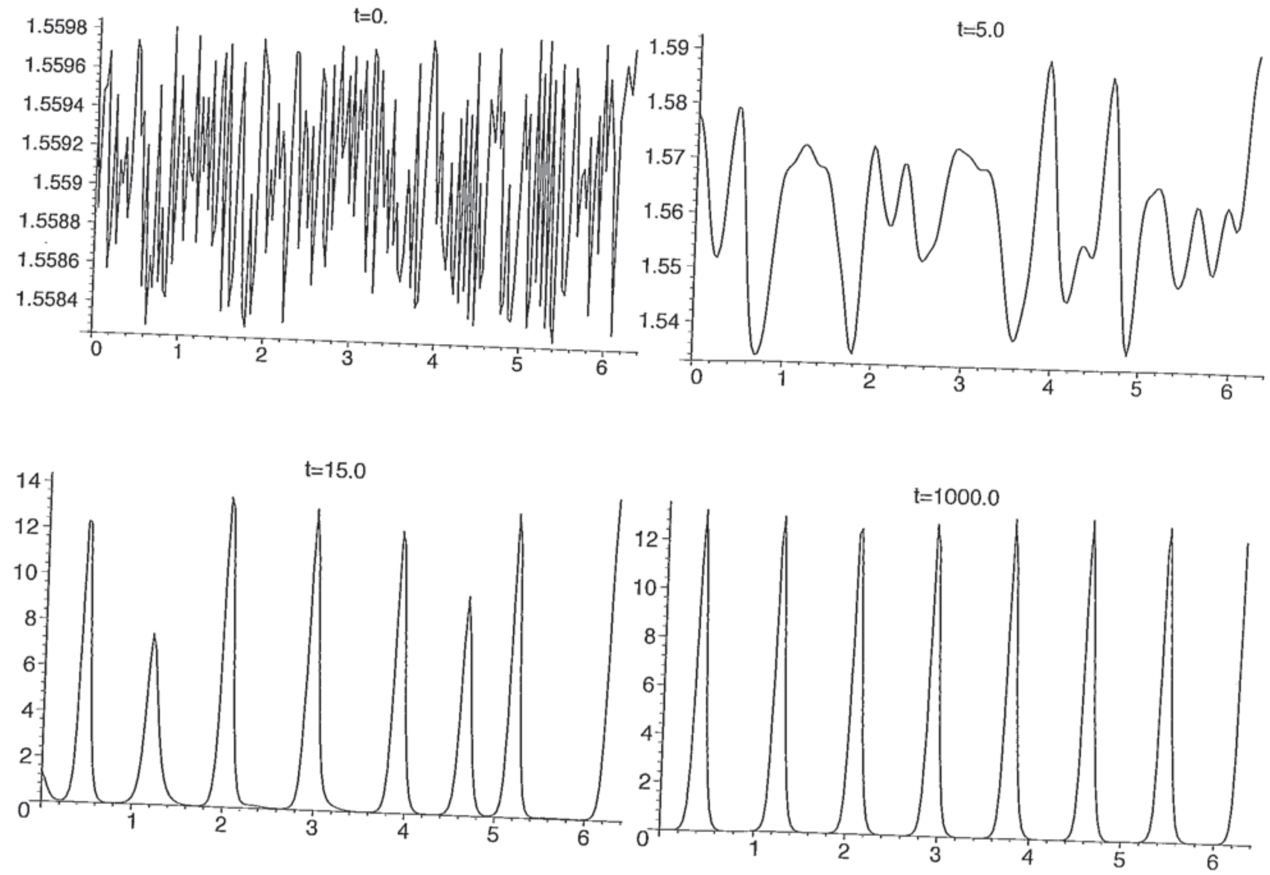
The endpoints of instability band are roots of $f(m)$.

The instability band exists iff $\frac{D_2}{D_1} > r$

Where r is given by:

$$\left(1+V_0^{+2}-r\right)^2 = 4r\left(V_0^{+2}-1\right)$$

In particular, $r=2$ if $A=2$, $V_0^+=1$ and r increases with A.



Grey-Scott model:

Example of Turing instability, with $A = 2.3$, $D_1 = 1$, $D_2 = 1000$. Note that the final steady state consists of spikes.

Weakly nonlinear Turing analysis

We found that $\exists\ D_2^*$ s.t. the homogeneous steady state is unstable when $D_2 > D_2^*$; the instability is of the form

$$u = u_0 + \eta \cos(m x) e^{\lambda t}, \quad \eta \ll 1$$

Question: • Can we find η as a function of D_2 ?
• What happens to a Turing pattern as $t \to$

Such questions can be answered when D_2 is near

Example: Consider the Ginzburg-Landau equation

$$\begin{cases} u_t = u_{xx} + \sigma u - u^3 , & x \in (0, \pi) \\ u(0) = 0 = u(\pi) \end{cases}$$

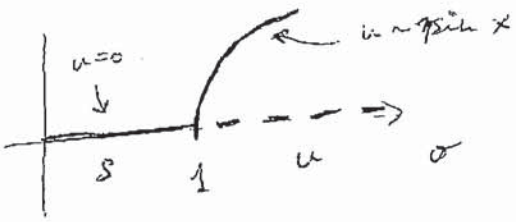
It has a steady state $u = 0$. To study its stability, we linearize:

$$u = 0 + \eta e^{\lambda t} \sin(mx) \quad , \eta \ll 1$$

Then $\quad \lambda = -m^2 + \sigma \qquad m = 1, 2 \ldots$

Note that $\lambda < 0 \quad \forall\, m = 1, 2, \ldots$ provided that $\sigma < 1$. As σ crosses 1, the ground state $u = 0$ becomes unstable, and

$u \sim \eta \sin(x)$, $\quad \eta \ll 1$.



Now suppose σ is <u>near</u> 1;

we want to find $\eta = \eta(\sigma)$. Suppose $\eta = \cancel{\delta}\, \varepsilon A$ and $\varepsilon \ll$

Ex **Rescale**: $u = \varepsilon U$

and let: $\sigma = 1 + \varepsilon^2$

$$\Rightarrow \begin{cases} U_t = U_{xx} + U + \varepsilon^2 (U - U^3) \\ U(0,t) = 0 = U(\pi, t) \end{cases}$$

Rescale time:

$$s = \varepsilon^2 t$$

$$\Rightarrow U_{xx} + U = \varepsilon^2 (U^3 - U + U_s)$$

Expand: $U = U_0(x,s) + \varepsilon^2 U_1(x,s) + \dots$

We get
$$\begin{cases} U_{0\,xx} + U_0 = 0 & (*) \\ U_{1\,xx} + U_1 = U_0^3 - U_0 + U_{0s} & (**) \end{cases}$$

with B.C. $U_{\substack{0\\1}}(0,s) = 0 = U_{\substack{0\\1}}(\pi, s)$

Thus $U_0(x,s) = A(s)\sin(t)$ where $A(s)$ is to be found.

Now:
$$\int_0^\pi (U_{1\,xx} + U_1) U_0 = \cancel{U_0 U_{1x}\Big|_0^\pi} - \cancel{U_{0x} U_1\Big|_0^\pi} + \int_0^\pi \underbrace{(U_{0\,xx} + U_0)}_{0} U_1$$
$$= 0$$

So multiply $(**)$ by U_0 & integrate to get:

$$\int_0^\pi U_0 (U_0^3 - U_0 + U_{0s}) = 0 \qquad (***)$$

Note:

- $$\int_0^{\pi} \sin^2(t)\,dt = \frac{\pi}{2}$$

- $$\int_0^{\pi} \sin^4(t)\,dt = \frac{3}{8}\pi$$

so that (***) becomes:

$$\frac{3}{8}\pi A^4 - \frac{\pi}{2} A^2 + \cancel{8}\frac{\pi}{2} A A_s = 0$$

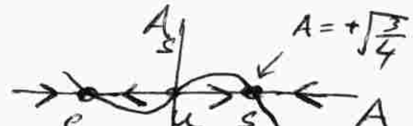
or

$$\boxed{A_s = A - \frac{3}{4} A^3} \qquad \star$$

- This eqn describes the slow evolution of the amplitude; remember that

$$\begin{cases} u(x,t) = \varepsilon A(\varepsilon^2 t)\sin(x) \\ \sigma = 1 + \varepsilon^2 \end{cases}$$

- Note that $A \to \pm\sqrt{\frac{3}{4}}$ as $s \to \infty$

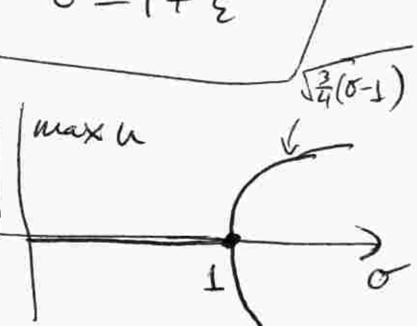


- Conclusion: for s.s. indep. of t, we get:

$$\boxed{u \sim \varepsilon\sqrt{\frac{3}{4}}\,\sin x, \quad \sigma = 1 + \varepsilon^2}$$

$$\max u \sim \varepsilon\sqrt{\frac{3}{4}}$$

$$\boxed{\max u \sim \sqrt{\frac{3}{4}(\sigma - 1)}, \quad \sigma > 1}$$



References

1) A. Turing, "The chemical basis of Morphogenesis", Phil. Trans. Royal Soc. B, 237 (1952)

2) J. Reynolds, "Complex patterns in a simple system", Science 261 (1993), p. 189-192.
(about Gray-scott model).

3) Holmes, M.H., "Introduction to Perturbation methods"
(see Chap. 6 for more on Ginzburg-Landau eq'n)

Some Questions:

1) For GS model (11), show that there is no Turing instability if $\frac{D_2}{D_1} < 2$.

2) Consider the Brusselator model:

$$u_t = D u_{xx} - u + \alpha + u^2 v$$
$$\tau v_t = D v_{xx} + (1-\beta) u - u^2 v$$

with $D, \tau, \alpha, \beta > 0$.

a) Find the steady states and classify their stability when $D = 0$.

b) Perform the Turing stability analysis of the steady states found in (a).

3) Consider the model of organic forming:

$$\begin{cases} u_t = u_{xx} + g(x) u - u^2, & x \in [0, \infty] \\ u'(0) = 0, \quad u'(\infty) = 0, & u \geq 0 \quad \forall\, x \geq 0. \end{cases}$$

with $g(x) = \begin{cases} 1, & x < l \\ -1, & x > l \end{cases}$.

Recall that if $l < \frac{\pi}{4}$ then the only sol'n is $u = 0$, but if $l > \frac{\pi}{4}$ then there is a non-zero solution. If $l = \frac{\pi}{4} + \varepsilon$ and $\varepsilon \ll 1$ then $u \sim A \varphi_0(x)$ where φ_0 solves

$$\begin{cases} -\varphi_0'' + g(x) \varphi_0 = 0 & \text{with } l = \frac{\pi}{4} \\ \varphi_0'(0) = 0, \quad \varphi_0'(\infty) = 0. \end{cases}$$

Determine A as a function of ε, $\varepsilon \ll 1$.