Ring and smoke-ring patterns in Gierer-Meinhardt system

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1 Introduction

The Gierer-Meinhardt model is a reaction-diffusion system:

$$\varepsilon^2 \Delta u - u + \frac{u^p}{v^q} = 0; \quad \Delta v - v + \frac{u^r}{v^s} = 0$$

with

$$\varepsilon \ll 1.$$

- Comes from mathematical biology (pattern formation in hydra)
- Very popular with mathematicians because it is non autonomous [no max principle, variational formulation] but still can be studied analytically.
- Simplest solutions are spikes; stability analysis very intricate [Doelman, Iron, Kaper, TK, Kowalczyk, Muratov, Ward, Winter, Wei];
- Many other solutions exist: asymmetric spikes [Doelman, Ward, Wei, Winter]
- Generalizations: heterogenous diffusion coefficients [Ward, Wei, Winter]; multiple activators/inhibitors [Wei, Winter]
- What about non-spiky solutions?
# 2 Ring solutions

Consider *radially symmetric* solutions of GM system inside a ball of radius $R$:

\[
\begin{align*}
\varepsilon^2 \left( u_{rr} + \frac{N-1}{r} u_r \right) - u + \frac{v^p}{v^q} &= 0; \\
v_{rr} + \frac{N-1}{r} v_r - v + \frac{1}{\varepsilon} \frac{u'}{v'} &= 0; \\
v'(0) = v'(R) = u'(0) = u'(R) = 0
\end{align*}
\]

We seek solutions that concentrate on a surface of a sphere of radius $r_0$. In 2-D they look like this:
**Theorem 1**: Let

\[ M_R(r) := \frac{1}{r} (N - 1) \frac{p - 1}{q} + \frac{J'_1(r)}{J_1(r)} + \frac{J'_2(r)}{J_{2,R}(r)}, \]  

where

\[ J_{2,R}(r) = J_2(r) - \frac{J'_2(R)}{J'_1(R)} J_1(r); \]

and \( J_1, J_2 \) satisfy

\[ J_{rr} + \frac{N - 1}{r} J_r - J = 0 \]

with

\[ J'_2(0) = 0; \quad J_1(r) \sim \ln(r) \quad \text{as} \quad r \to 0. \]

Suppose that \( r_0 \) satisfies

\[ M_R(r_0) = 0. \]

Then there exists a ring-type solution concentrated at the radius \( r = r_0 \), of the form

\[ u(x) \sim C w \left( \frac{|x| - r_0}{\varepsilon} \right), \quad \varepsilon \to 0 \]
where $C$ is some constant and $w$ is the ground state

$$w_{yy} - w + w^p = 0; \quad w \sim Ce^{-|y|}, \quad y \to \infty.$$  

Remark:

$$J_1(r) = r^{2-N} I_\nu(r), \quad J_2(r) = r^{2-N} K_\nu(r), \quad \nu = \frac{N-2}{2}$$

where $I_\nu, K_\nu$ are modified Bessel functions of order $\nu$.

Remark: In the case of $N=3$, $J_1, J_2$ can be computed explicitly:

$$J_1 = \frac{\sinh r}{r}, \quad J_2(r) = \frac{e^{-r}}{4\pi r}. \quad (2)$$
**Proof (Standard GM in 2d):** In radial variables:

\[ \varepsilon^2 u_{rr} + \varepsilon^2 \frac{1}{r} u_r - u + \frac{u^2}{v} = 0; \quad v_{rr} + \frac{1}{r} v_r - \frac{u^2}{\varepsilon} = 0 \]

**Inner problem:**

\[ r = r_0 + \varepsilon y; \]
\[ u = U_0(y) + \varepsilon U_1(y) + \cdots; \quad v = V_0 + V_1(y) + \cdots \]

**Leading order:**

\[ 0 = U_{0yy} - U_0 + \frac{U_0^2}{V_0}; \quad V_{0yy} = 0 \]
\[ U_0(y) = \xi w(y); \quad w_{yy} - w + w^2 = 0; \]
\[ V_0(y) = \xi; \quad \text{(to be determined later)} \]

**O(\varepsilon) terms:**

\[ 0 = U_{1yy} - U_1 + 2 \frac{U_0}{V_0} U_1 - \frac{U_0^2}{V_0^2} V_1 + \frac{1}{r_0} U_{0y}; \quad V_{1yy} + U_0^2 = 0 \]
\( V_{1y} = - \int_0^y U_0^2 \bigg|_{\text{odd}} + A \) \hspace{1cm} (3)

\( U_{1yy} - U_1 + 2wU_1 = w^2V_1 - \frac{1}{r_0} w_y \) \hspace{1cm} (4)

Multiply (4) by \( w_y \) and integrate by parts:

\[ 0 = \int w_y w^2 V_1 - \frac{1}{r_0} \int w_y^2 \]

\[ \int w_y w^2 V_1 = -A \int \frac{w^3}{3} \]

\[ r_0 = -\frac{\xi}{A} \int \frac{w_y^2}{3}. \]
To determine $\xi$, $A$ we look at the **outer problem**:

$$v_{rr} + \frac{1}{r} v_r - v = -\frac{u^2}{\varepsilon}; \quad \frac{u^2}{\varepsilon} \sim C \delta(r - r_0); \quad C = \xi^2 \left( \int w^2 \, dy \right) = 6\xi^2.$$  

$$v = 6\xi^2 G(r, r_0) \quad \text{where} \quad G_R(r, r_0) = \frac{1}{J_1'(r_0) J_{2,R}'(r_0) - J_1(r_0) J_{2,R}'(r_0)} \left\{ \begin{array}{ll} J_{2,R}(r_0) J_1(r), & \text{if } r < r_0, \\ J_1(r_0) J_{2,R}(r), & \text{if } r > r_0. \end{array} \right.$$

**Matching**: In inner variables:

\[
\begin{align*}
    r &= r_0 + \varepsilon y; \\
    v &= 6\xi^2 G(r_0^+, r_0) + \varepsilon y 6\xi^2 \left\{ \begin{array}{ll} G_r(r_0^+, r_0), & \text{if } y > 0 \\ G_r(r_0^+, r_0), & \text{if } y < 0 \end{array} \right. \\
    \xi &= 6\xi^2 G(r_0, r_0) \\
-\xi^2 + A &= 6\xi^2 G_r(r_0^+, r_0); \quad \xi^2 + A = 6\xi^2 G_r(r_0^-, r_0) \\
A &= 6\xi^2 \left( G_r(r_0^+, r_0) + G_r(r_0^-, r_0) \right)
\end{align*}
\]

Finally,

\[
\frac{1}{r_0} + \frac{J_1'(r_0)}{J_1(r_0)} + \frac{J_{2,R}'(r_0)}{J_{2,R}(r_0)} = 0.
\]
Theorem 2: Let

\[ a = (N - 1)^{p - 1} \frac{p - 1}{q} \]

Suppose that \( N \geq 3 \). There are three cases.

1.a) If \( N - 2 < a < N - 1 \) then there exists \( R_0 \) such that if \( R > R_0 \) then \( M_R(r) = 0 \) has exactly two solutions \( 0 < r_1 < r_2 < R \), and if \( R < R_0 \), then \( M_R(r) = 0 \) has no solution. Moreover, for \( R > R_0 \), \( M'_R(r_1) < 0, M'_R(r_2) > 0 \).

1.b) If \( a \geq N - 1 \) then \( M_R(r) = 0 \) has no solution for any \( R \).

1.c) If \( a \leq N - 2 \) then \( M_R(r) = 0 \) has precisely one solution \( r_1 \) for any \( R \) and moreover \( M'_R(r_1) > 0 \).

Suppose that \( N = 2 \). Then there exists a number \( a_\infty > 1 \) whose numerical value is \( a_\infty = 1.06119 \) such that one of the following holds:

2.a) If \( a \in (0, a_\infty) \) then the situation is the same as in case (1.a).

2.b) If \( a > a_\infty \) then \( M_R(r) > 0 \) for any \( R \).

2.c) If \( a = a_\infty \) then \( M_R(r) > 0 \) any \( R < \infty \). When \( R = \infty \), there exists a number \( r_0 \) such that \( M_\infty(r_0) = 0 = M'_\infty(r_0) \), and \( M_R(r) > 0 \) for any \( r \neq r_0 \).

As the statement indicates, the situation for \( N = 2 \) is very different from \( N \geq 3 \). The case \( N = 2 \) and \( a \in (1, a_\infty) \) has no analogue in higher dimensions and is considerably more difficult.
Sketch of proof \((N \geq 3)\) Here is \(M_R\) for several \(R\) values:

\[
\begin{align*}
\text{Step 1. } & \ M_R \text{ is positive for small } R. \text{ For small } R, \text{ expand} \\
& \quad rM_R(r) \sim a - \left( \frac{1 - r_0^N}{\frac{1}{N-2} + N \frac{r_0^{N-2}}{R^2}} \right), \quad r_0 = \frac{r}{R} \in (0, 1); \quad R \ll 1
\end{align*}
\]

rhs is \(a - N + 2\) when \(r = 0\) and increases from there (hence never crosses 0)

\textbf{Step 2.} Since \(J_{2,R}^I(R) = 0\), it follows that \(M_R(R) = \frac{a}{R} + \frac{J_{1}^I(R)}{J_{1}(R)}\). But \(J_1\) is a strictly increasing and positive function so that \(M_R(R)\) is always strictly positive.
Step 3. $M_R(r)$ has a double root iff

$$r(J_1(r)J'_{2,R}(r))' = -aJ_1(r)J_{2,R}(r); \quad \frac{J'_{2,R}(r)}{J_{2,R}(r)} = -\frac{a}{r} - \frac{J'_1(r)}{J_1(r)}$$

Eliminate $R$ to get:

$$g(r) := (a^2 - a(N - 2) - 2r^2)J_1^2(r) + 2raJ_1(r)J'_1(r) + 2r^2J'_1^2(r) = 0.$$  \hspace{1cm} (7)

$g(r)$ satisfies

$$rg' + r^2Cg = J_1^2(r)(B - Ar^2)$$

with

$$A = 4(N - 1 - a), \quad B = (2N - a - 4)(a + 2 - N)a, \quad C = 2N - 4 - a$$

Moreover

$$g(0) = a(a - N + 2) > 0; \quad g(\infty) \rightarrow -\infty$$

so $g$ has at least one root. Let $r_1$ be the first root of $g$. then $g'(r_1) < 0$ so rhs(8)<0. But rhs changes sign only once (and is negative for $r > r_1$); so $g$ cannot have any more roots.

Step 4. For sufficiently large $R$, $M_R$ has a single root (due to large-argument expansion of $M_\infty(r)$).
The situation is *more complicated* for $N = 2$. Difficult theorem:

- $M_\infty(r)$ has *exactly* 1 root if $0 < a < 1$
- $M_\infty(r)$ has *exactly* 2 roots if $1 < a < a_c = 1.06$
- $M_\infty(r)$ has *no* roots if $a > a_c$.

When $1 \ll R \ll \infty$, there is a sharp transition of $r_0$ as $a$ crosses 1.
3 Smoke-ring solutions

Consider GM in all of $\mathbb{R}^3$; we seek solutions that concentrate on a ring. By taking a cross-section in cylindrical coordinates, this becomes a 2-D problem in $(r, z)$ space:
Define the logarithmic scale:

\[ \eta = \frac{-1}{\ln \varepsilon}. \]

Note that we have the relationship

\[ 0 \ll \varepsilon \ll \eta \ll 1. \] (10)

After proper scaling, the standard GM system is:

\[ 0 = \varepsilon^2 \left( \frac{\Delta_{(r,z)} u}{r} + \frac{1}{r} u_r \right) - u + \frac{u^2}{v}; \quad 0 = \left( \frac{\Delta_{(r,z)} v}{r} + \frac{1}{r} u_r \right) - v + \frac{\eta}{\varepsilon^2} u^2 \] (11)

**Outer problem:** \( u \) is a spike at \( x_0 = (r_0, z_0) \) so we estimate

\[ \frac{\eta}{\varepsilon^2} u^2 \sim C \delta(x - z_0), \quad \text{where} \quad C = \int \frac{\eta}{\varepsilon^2} u^2 \, dx \]

So

\[ u = CG(x, x_0) \]

where \( G \) is the Green’s function which satisfies:

\[ \Delta G + \frac{1}{r} G_r - G = -\delta(x, x_0). \]
Descent from 3D: $G$ is a convolution of the 3D Green’s function along a ring of radius $r_0$:

$$G(x, x_0) = \int_{R^3} \frac{e^{-|x-x'|}}{4\pi |x - x'|} R(x') dx'$$

where $R(x')$ is the ring of 2d delta functions:

$$G(r, z, r_0, z_0) = \frac{r_0}{4\pi} \int_0^{2\pi} \frac{\exp[-(r^2 + r_0^2 - 2rr_0 \cos \omega + (z - z_0)^2)^{1/2}]}{4\pi(r^2 + r_0^2 - 2rr_0 \cos \omega + (z - z_0)^2)^{1/2}} d\omega$$

(12)

After change of variables we have:

$$G = \frac{r_0 e^{-\beta}}{\pi(\alpha - \beta)} \int_0^1 \frac{\exp[-(\alpha - \beta)\tau]}{\sqrt{\tau(\delta + \tau)(1 + \delta + \tau)(1 - \tau)}} d\tau,$$

where

$$\beta = [(r - r_0)^2 + (z - z_0)^2]^{1/2}; \quad \alpha = [(r + r_0)^2 + (z - z_0)^2]^{1/2}; \quad \delta = \frac{2\beta}{\alpha - \beta} \ll 1;$$
After 7 pages of complicated computations we get the following expansion.

\[
G(x_0 + \varepsilon y, x_0) = \frac{1}{2\pi \eta} \left[ 1 - \eta \ln R + \eta F_0 + \frac{\varepsilon \rho}{2r_0} (-1 + \eta \ln R + \eta F_1(r_0)) + O(\varepsilon^2) \right]
\]

where

\[
y = \frac{x - x_0}{\varepsilon} = (\rho, Z); \quad R = |y| = \sqrt{\rho^2 + Z^2}; \quad \eta = \frac{1}{\ln (1/\varepsilon)}; \quad (13)
\]

\[
F_0(r_0) = g_1(2r_0) + \ln 4r_0 \quad \text{where} \quad g_1(2r_0) = \int \left( \frac{\exp(-2r_0\tau)}{\tau \sqrt{1 - \tau^2}} - \frac{1}{\tau} \right) d\tau \quad (14)
\]

\[
F_1(r_0) = 2r_0g_1'(2r_0) - g_1(2r_0) - \ln 4r_0 + 1 \quad (15)
\]

The outer solution in the inner variables becomes:

\[
v \sim \xi \left[ 1 - \eta \ln R + \eta F_0 + \frac{\varepsilon \rho}{2r_0} (-1 + \eta \ln R + \eta F_1(r_0)) \right], \quad |y| \to \infty
\]

where \( \xi \) is given by

\[
2\pi \xi = \int \frac{u^2}{\varepsilon^2} dx.
\]

The smoke-ring radius \( r_0 \) will be determined by \( O(\varepsilon \eta) \)!! This requires an expanding

\[
\xi = \xi_{00} + \eta \xi_{01} + O(\varepsilon).
\]
**Inner problem,** $y = \frac{x - x_0}{\varepsilon}$:

Expand in $\varepsilon$ while treating $\eta$ as a constant:

\[
\begin{align*}
    u(x, t) &= U = U_0(|y|) + \varepsilon U_1(y) + \cdots \\
    V(x, t) &= V = V_0(|y|) + \varepsilon V_1(y) + \cdots \\
    \xi &= \xi_0 + \varepsilon \xi_1 + \cdots
\end{align*}
\]

The leading order equations are

\[
\begin{align*}
    0 &= \Delta U_0 - U_0 + \frac{U_0^2}{V_0} \\
    0 &= \Delta V_0 + \eta U_0^2 \\
    2\pi \xi_0 &= \int U_0^2 dx.
\end{align*}
\]

Next we expand in $\eta$:

\[
\begin{align*}
    U_0 &= U_{01} + \eta U_{11}; \\
    V_0 &= V_{01} + \eta V_{11}; \\
    \xi_0 &= \xi_{01} + \eta \xi_{01}.
\end{align*}
\]

We get

\[
\xi_{00} = \frac{1}{\int_0^\infty w^2(s)sds} = 0.20266 \quad \text{where} \quad \Delta w - w + w^2 = 0
\]
After 2 pages of computation,

\[ \xi_{01} = \xi_{00} (\alpha - 2F_0); \quad \alpha = 0.3833 \]

This gives us a correction to \( O(\varepsilon \eta) \) term:

\[
v \sim (\xi_{00} + \eta \xi_{01}) \left( 1 - \eta \ln R + \eta F_0 + \frac{\varepsilon \rho}{2r_0} (-1 + \eta \ln R + \eta F_1) \right) \\
= \xi_{00} \left( 1 - \eta \ln R - \eta (F_0 + \alpha) + \frac{\varepsilon \rho}{2r_0} (-1 + \eta \ln R + \eta (F_1 + 2F_0 - \alpha)) \right)
\]

Next we must study the \( O(\varepsilon + \varepsilon \eta) \) terms... After 4 more pages of solvability computations involving adjoint operator... we finally get the equation for \( r_0 \):

\[
F_1(r_0) + 2F_0(r_0) - \alpha + \beta = 0, \quad \text{where} \quad \alpha = 0.3833, \quad \beta = 0.087
\]