

# The Transition to a Point Constraint in a Mixed Biharmonic Eigenvalue Problem

A. E. LINDSAY, M. J. WARD, T. KOLOKOLNIKOV.

Alan Lindsay, Dept. of Applied and Computational Mathematics and Statistics, Univ. of Notre Dame, South Bend, Indiana, 46656, USA,

Theodore Kolokolnikov; Dept. of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada,

Michael Ward; Dept. of Mathematics, Univ. of British Columbia, Vancouver, B.C., V6T 1Z2, Canada.

(Received 25 July 2014)

The mixed-order eigenvalue problem  $-\delta\Delta^2u + \Delta u + \lambda u = 0$  with  $\delta > 0$ , modeling small amplitude vibrations of a thin plate, is analyzed in a bounded 2-D domain  $\Omega$  that contains a single small hole of radius  $\varepsilon$  centered at some  $x_0 \in \Omega$ . Clamped conditions are imposed on the boundary of  $\Omega$  and on the boundary of the small hole. In the limit  $\varepsilon \rightarrow 0$ , and for  $\delta = \mathcal{O}(1)$ , the limiting problem for  $u$  must satisfy the additional point constraint  $u(x_0) = 0$ . To determine how the eigenvalues of the Laplacian in a domain with a small hole are perturbed by adding the small fourth order term  $-\delta\Delta^2u$ , together with an additional boundary condition on  $\partial\Omega$  and on the hole boundary, the asymptotic behavior of the eigenvalues of the mixed-order eigenvalue problem are studied in the dual limit  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow 0$ . Three ranges of  $\delta \ll 1$  are uncovered and analyzed:  $\delta = \mathcal{O}(\varepsilon^2)$ ,  $\delta \ll \mathcal{O}(\varepsilon^2)$ , and  $\mathcal{O}(\varepsilon^2) \ll \delta \ll 1$ . In the regime  $\mathcal{O}(\varepsilon^2) \ll \delta \ll 1$  it is shown that the leading-order asymptotic behavior of an eigenvalue of the mixed-order eigenvalue problem is asymptotically independent of  $\varepsilon$ . Therefore, it is this regime that provides a transition to the point constraint behavior characteristic of the range  $\delta = \mathcal{O}(1)$ . The asymptotic results for the eigenvalues are validated by full numerical simulations of the PDE.

## 1 Introduction

The determination of eigenfrequencies and eigenmodes characterizing the small amplitude vibration of thin plates is an important problem in mechanics. In the framework of Kirchhoff-Love plate theory (cf. [21]), an eigenmode of vibration, characterizing the out-of-plane deflection  $w$  of the plate, is a nontrivial solution of

$$-D\Delta^2w + T\Delta w + \rho h\omega^2w = 0, \quad x \in \Omega \subset \mathbb{R}^2; \quad w = \partial_n w = 0, \quad x \in \partial\Omega, \quad (1.1)$$

that occurs for certain discrete values of  $\omega$ . Here  $\rho$  is the material density,  $D = Eh^3/[12(1-\mu^2)]$  is the flexural rigidity of the plate defined in terms of Young's modulus  $E$ , the plate thickness  $h$ , and the Poisson's ratio  $\mu$ , while  $T$  is the in-plane tension applied at the edges of the plate. To understand the physical significance of each of the terms in (1.1), it is useful to consider it as the Euler-Lagrange equation of the energy functional

$$\mathcal{E}[\phi] \equiv \int_{\Omega} \left[ \frac{1}{2}\rho h \left( \frac{\partial\phi}{\partial t} \right)^2 + \frac{1}{2}D (\Delta\phi)^2 + \frac{1}{2}T |\nabla\phi|^2 \right] dx, \quad (1.2)$$

after applying the ansatz  $\phi(x, t) = w(x)e^{i\omega t}$ . The three terms in the integrand of (1.2) correspond to the kinetic, bending, and stretching, energies of the plate, respectively. In the present work, the density  $\rho$ , the thickness  $h$ , the flexural rigidity  $D$ , and the tension  $T$ , are taken to be spatially uniform positive constants.

The eigenproblem (1.1), augmented by inserting small clamped holes, can be re-cast in a more convenient dimen-

sionless form as

$$-\delta\Delta^2 u_\varepsilon + \Delta u_\varepsilon + \lambda_\varepsilon u_\varepsilon = 0, \quad x \in \Omega \setminus \Omega_\varepsilon; \quad \int_{\Omega \setminus \Omega_\varepsilon} u_\varepsilon^2 dx = 1, \quad (1.3 a)$$

$$u_\varepsilon = \partial_n u_\varepsilon = 0, \quad x \in \partial\Omega; \quad u_\varepsilon = \partial_n u_\varepsilon = 0, \quad x \in \partial\Omega_\varepsilon. \quad (1.3 b)$$

Here  $\Omega$  is a bounded domain in  $\mathbb{R}^2$ , and  $\Omega_\varepsilon$  is a collection of  $N$  small non-overlapping holes with centers  $x_j \in \Omega$ , for  $j = 0, \dots, N-1$ , for which the  $j$ -th hole shrinks uniformly to a point  $x_j \in \Omega$  as  $\varepsilon \rightarrow 0$ . A schematic representation of the perturbed domain  $\Omega \setminus \Omega_\varepsilon$  is shown in Fig. 1. In (1.3), the key parameter  $\delta$ , which reflects the relative importance of the bending and stretching energies, and a new eigenvalue parameter  $\lambda_\varepsilon$  are defined by

$$\delta \equiv \frac{D}{T}, \quad \lambda_\varepsilon \equiv \frac{\rho h \omega^2}{T}. \quad (1.4)$$

The eigenvalues  $\lambda_\varepsilon$  of (1.3) determine the frequencies of vibrations of the perforated thin plate by  $\omega = \sqrt{T\lambda_\varepsilon/\rho h}$ .

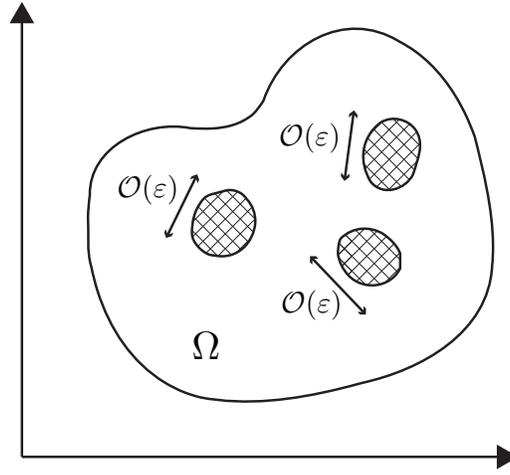


Figure 1. Schematic diagram of a perturbed region  $\Omega \setminus \Omega_\varepsilon$  consisting of three small holes.

Perforated plate structures are commonly used in many engineering design systems such as heat exchangers in nuclear power systems, sound absorbing screens, or pressure vessels (cf. [7], [4], [16]). In engineering design, drilling a small hole inside a plate is generally the easiest method to alter an undesirable natural frequency of a plate structure without incurring any significant degradation in the structural integrity of the plate (cf. [4], [16]). Effective medium theories, with varying degrees of success, have been used to model the effect of perforations on either the bending of a rectangular plate (cf. [3]) or on the resonant frequencies of a plate (cf. [7], [4]). The development of methodologies to numerically compute resonant frequencies of plates, with and without perforations, are described in [2], [6], and [16].

In contrast, in this paper we will use the method of matched asymptotic expansions to analyze (1.3) in the dual limit  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow 0$ . Before describing our work in detail, we set the context of our study by first briefly summarizing some key previous mathematical results for singularly perturbed biharmonic eigenvalue problems.

From a mathematical viewpoint, the spectrum of the pure biharmonic operator is known to have some rather surprising features owing to the lack of a maximum principle. In a domain with a single small hole  $\Omega_\varepsilon$  of radius  $\varepsilon$  centered at some  $x_0 \in \partial\Omega$ , the (pure) biharmonic eigenvalue problem is formulated as

$$\Delta^2 u = \lambda u, \quad x \in \Omega \setminus \Omega_\varepsilon; \quad u = \partial_n u = 0, \quad x \in \partial\Omega; \quad (1.5 a)$$

$$u = \partial_n u = 0, \quad x \in \partial\Omega_\varepsilon; \quad \int_{\Omega \setminus \Omega_\varepsilon} u^2 dx = 1. \quad (1.5 b)$$

This system can be obtained formally from (1.3) by setting  $N = 1$  and taking the limit  $\delta \rightarrow +\infty$  while replacing  $\lambda$  by  $\delta\lambda$ . For the annular domain  $0 < \varepsilon \leq |x| \leq 1$  in 2-D, it was shown in [8] and [9] that the principal eigenfunction for

(1.5) is not radially symmetric when  $\varepsilon$  is below a critical value. More general results showing that the fundamental mode of vibration for the biharmonic operator is not necessarily of one sign are given in (cf. [22]).

Another key qualitative feature of the spectrum of (1.5), as described in [5], is that the limiting behavior of a simple eigenvalue  $\lambda$  of (1.5) is given by

$$\lambda = \lambda_0 + 4\pi|\nabla u_0(x_0)|^2\nu_b + \mathcal{O}(\nu_b^2), \quad \nu_b \equiv -\frac{1}{\log \varepsilon}, \quad (1.6)$$

where  $(u_0, \lambda_0)$  is an eigenpair of the associated *point constraint problem*

$$\Delta^2 u_0 = \lambda_0 u_0, \quad x \in \Omega \setminus \{x_0\}; \quad u_0 = \partial_n u_0 = 0, \quad x \in \partial\Omega; \quad \int_{\Omega} u_0^2 dx = 1; \quad u_0(x_0) = 0. \quad (1.7)$$

A remarkable feature of the limiting problem (1.7) is that, due to the additional *point-constraint* requirement  $u_0(x_0) = 0$  in (1.7), it is *not* the problem in the absence of a perturbing hole. This result implies that no matter how small  $\varepsilon$  is made, the vibrational frequencies of the perturbed plate do not approach those of a defect free plate. In [5] and [15] asymptotic expansions for  $\lambda$  were derived for (1.5) to capture higher order  $\varepsilon$ -dependent correction terms to (1.6) for both the generic situation where  $|\nabla u_0(x_0)| \neq 0$  and for the degenerate case where  $|\nabla u_0(x_0)| = 0$ . Point constraints also arise in the construction of solutions to nonlinear eigenvalue problems  $\Delta^2 u = \lambda f(u)$  in two dimensions [17, 18].

This limiting point constraint behavior for (1.5) as  $\varepsilon \rightarrow 0$  is qualitatively very different from the well-known results (cf. [19], [12], [14], [23], [24]) for the asymptotic behavior of eigenvalues of the Laplacian in the limit of asymptotically small hole radius  $\varepsilon \rightarrow 0$ , formulated as

$$\Delta u + \lambda u = 0, \quad x \in \Omega \setminus \Omega_\varepsilon; \quad \int_{\Omega \setminus \Omega_\varepsilon} u^2 dx = 1; \quad (1.8 a)$$

$$u = 0, \quad x \in \partial\Omega; \quad u = 0, \quad x \in \partial\Omega_\varepsilon. \quad (1.8 b)$$

For the case of a single hole centered at  $x_0$ , it was shown in [23] that the asymptotic behavior of a simple eigenvalue of (1.8) is

$$\lambda \sim \lambda^*(\nu) + \mathcal{O}(\varepsilon\nu), \quad \text{where} \quad \nu \equiv -1/\log(\varepsilon d), \quad (1.9)$$

and  $d$  is the logarithmic capacitance (cf. [20]) associated with the hole. Here the function  $\lambda^*(\nu)$ , which has the effect of summing an infinite logarithmic series in powers of  $\nu$ , satisfies a transcendental equation involving the regular part of the Green's function for the Helmholtz operator. As  $\nu \rightarrow 0$ , then  $\lambda^* \rightarrow \lambda_0$ , where  $\lambda_0$  is a simple eigenvalue of the following limiting problem in the absence of a hole:

$$\Delta u_0 + \lambda_0 u_0 = 0, \quad x \in \Omega; \quad \int_{\Omega} u_0^2 dx = 1; \quad u_0 = 0, \quad x \in \partial\Omega. \quad (1.10)$$

With this background, the goal of this paper is to investigate the mixed-order eigenvalue problem (1.3) in the limit  $\varepsilon \rightarrow 0$  for various ranges of the parameter  $\delta > 0$ , measuring the relative strength of the fourth order term. For simplicity, we will only consider the case of a single hole where  $N = 1$ . For  $\delta = \mathcal{O}(1)$ , and in the limit  $\varepsilon \rightarrow 0$  of small hole radius, an eigenvalue of the perturbed problem (1.3) tends to an eigenvalue of the corresponding point constraint problem associated with (1.3) (see (3.1) below). However, the previous analyses of (1.5) and (1.8), as described above, suggest a dichotomy of possible behaviors associated with (1.3) in the dual limit  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow 0$ . If  $\delta$  is small enough relative to  $\varepsilon$ , then the limiting problem as  $\varepsilon \rightarrow 0$  should be the problem with no hole, whereas if  $\delta$  is large enough (relative to  $\varepsilon$ ) the limiting problem as  $\varepsilon \rightarrow 0$  should be one with a point constraint. The goal of this paper is to study the transition between these two cases as  $\delta \ll 1$  is varied.

Our analysis on (1.3) reveals that there are three distinguished regimes in the  $\varepsilon, \delta$  plane as  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow 0$  where different eigenvalue asymptotics occur. For the regime  $\delta \ll \mathcal{O}(\varepsilon^2)$ , the leading-order eigenvalue asymptotics for (1.3) is essentially the same as that for the Laplacian, as given in (1.9) (see Principal Result 4.3 below). For the distinguished limit  $\delta = \mathcal{O}(\varepsilon^2)$ , the leading-order eigenvalue asymptotics as  $\varepsilon \rightarrow 0$  is again given by (1.9), but where  $d$  is replaced by a new quantity  $d(\delta_0)$ , where  $\delta_0 \equiv \delta/\varepsilon^2 = \mathcal{O}(1)$ . Here  $d(\delta_0)$  is determined from the far-field behavior of a canonical fourth-order problem defined near the hole (see Principal Result 4.2 below). The third regime is where  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$ .

In this regime we show in Principal Result 4.4 that the leading-order asymptotics of an eigenvalue of (1.3) is given by (1.9), but where  $\varepsilon d$  is replaced by  $2\sqrt{\delta}e^{-\gamma\varepsilon}$ . Since the resulting leading-order eigenvalue asymptotics is independent of the hole radius, it is this regime that provides a transition to the point constraint behavior characteristic of the  $\delta = \mathcal{O}(1)$  regime. Finally, in Principal Result 4.5 we improve this leading-order eigenvalue approximation by adding to it a transcendently small term of order  $\mathcal{O}(\sqrt{\delta})$  that results from analyzing the boundary layer near  $\partial\Omega$ .

The outline of this paper is as follows. In §2 we study the exactly solvable problem (1.5) in an annular domain in order to clearly motivate the necessity of a point constraint for the limiting solution as  $\varepsilon \rightarrow 0$ . In §2.1 we formulate and analyze an exactly solvable model biharmonic BVP in an annulus so as to motivate the various scalings in  $\delta$  and  $\varepsilon$  that arise in the asymptotic analysis of (1.3). In §3 we analyze the  $\delta \rightarrow 0$  limit of the solution to the point constraint problem associated with (1.3) for a single hole. In §4.2–4.4, which constitutes the main focus of this paper, we will analyze (1.3) for a single hole in the limit  $\varepsilon \rightarrow 0$  for the three asymptotic ranges of  $\delta \ll 1$  given by  $\delta = \mathcal{O}(\varepsilon^2)$ ,  $\delta \ll \mathcal{O}(\varepsilon^2)$ , and  $\mathcal{O}(\varepsilon^2) \ll \delta \ll 1$ . For  $\delta \ll 1$ , the effect of the boundary layer along  $\partial\Omega$  on the eigenvalue asymptotics is also examined. In §4.4 and §5 we validate our asymptotic theory for the regime  $\mathcal{O}(\varepsilon^2) \ll \delta \ll 1$  with full numerical PDE computations of (1.3). Finally, in §6 we suggest a few open problems that warrant further study.

## 2 Two Exactly Solvable Model Problems

To clarify the requirement of a point constraint in the limiting problem of (1.5) as  $\varepsilon \rightarrow 0$ , it is useful to consider the special case where  $\Omega = \{x \mid |x| \leq 1\}$ ,  $x_0 = (0, 0)$ , and  $\Omega_\varepsilon = \{x \mid |x| \leq \varepsilon\}$  in which closed form solutions are obtainable. To obtain exact radially symmetric solutions of (1.5) for the annulus, we factor  $\Delta^2 u - \lambda u = (\Delta - \mu^2)(\Delta + \mu^2)u = 0$ , where  $\mu \equiv \lambda^{1/4}$ , so that the separable radially symmetric eigenfunctions are spanned by  $\{J_0(\mu r), I_0(\mu r), K_0(\mu r), Y_0(\mu r)\}$ . For the  $\varepsilon = 0$  problem, with no perturbing hole at the origin, the radially symmetric eigenfunctions  $u_*(r)$  with smooth behavior at the origin are given by

$$u_*(r) = A \left[ J_0(\mu_* r) - \frac{J_0(\mu_*)}{I_0(\mu_*)} I_0(\mu_* r) \right], \quad \text{where} \quad I_1(\mu_*) J_0(\mu_*) + I_0(\mu_*) J_1(\mu_*) = 0. \quad (2.1)$$

For the annulus, where  $\varepsilon > 0$ , an exact radially symmetric solution  $u_\varepsilon(r)$  is constructed from a linear combination of the set  $\{J_0(\mu r), I_0(\mu r), K_0(\mu r), Y_0(\mu r)\}$ , such that  $u_\varepsilon = \partial_r u_\varepsilon = 0$  on  $r = \varepsilon$  and  $r = 1$ . The eigenvalues of the system, with  $\lambda_\varepsilon^{1/4} = \mu_\varepsilon$ , are determined by the condition

$$\det \begin{bmatrix} J_0(\mu_\varepsilon) & I_0(\mu_\varepsilon) & K_0(\mu_\varepsilon) & Y_0(\mu_\varepsilon) \\ J_1(\mu_\varepsilon) & -I_1(\mu_\varepsilon) & K_1(\mu_\varepsilon) & Y_1(\mu_\varepsilon) \\ J_0(\varepsilon\mu_\varepsilon) & I_0(\varepsilon\mu_\varepsilon) & K_0(\varepsilon\mu_\varepsilon) & Y_0(\varepsilon\mu_\varepsilon) \\ J_1(\varepsilon\mu_\varepsilon) & -I_1(\varepsilon\mu_\varepsilon) & K_1(\varepsilon\mu_\varepsilon) & Y_1(\varepsilon\mu_\varepsilon) \end{bmatrix} = 0. \quad (2.2)$$

In Fig. 2 the lowest eigenvalue  $\mu_{\varepsilon,0}$ , determined from numerical solution of (2.2), is plotted for a range of values of  $\varepsilon$ . We observe that the limiting behavior of  $\mu_\varepsilon$  as  $\varepsilon \rightarrow 0$  is not to an eigenvalue of (2.1) for the problem with no hole. Instead, as shown in [5] and [15], the limiting problem as  $\varepsilon \rightarrow 0$  requires a point constraint  $u_0(0) = 0$ , as specified in (1.7). The physical interpretation of this result is that a defect of spatial extent  $\mathcal{O}(\varepsilon)$ , for any  $\varepsilon > 0$ , will in general generate an  $\mathcal{O}(1)$  jump in the eigenvalue and the vibrational frequencies of the plate.

In the present example of a disk with a puncture at the center, we can construct the radially symmetric eigensolutions of this limiting problem satisfying  $u_0(0) = 0$  as

$$u_0 = A \left[ J_0(\mu_0 r) - I_0(\mu_0 r) - \left( \frac{J_0(\mu_0) - I_0(\mu_0)}{\frac{2}{\pi} K_0(\mu_0) + Y_0(\mu_0)} \right) \left( \frac{2}{\pi} K_0(\mu_0 r) + Y_0(\mu_0 r) \right) \right]. \quad (2.3 a)$$

Here  $A$  is a normalization constant, and  $\mu_0$  is a root of the eigenvalue equation

$$(J_0(\mu_0) - I_0(\mu_0)) \left( \frac{2}{\pi} K_1(\mu_0) + Y_1(\mu_0) \right) = (J_1(\mu_0) + I_1(\mu_0)) \left( \frac{2}{\pi} K_0(\mu_0) + Y_0(\mu_0) \right). \quad (2.3 b)$$

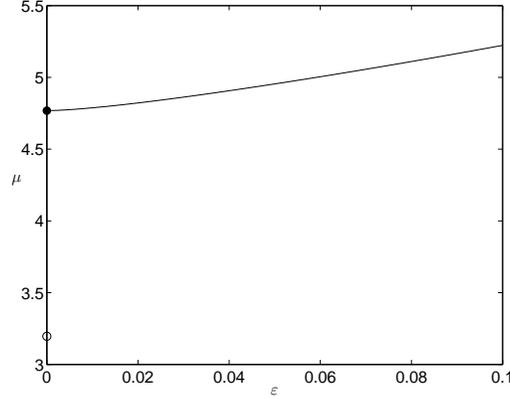


Figure 2. The lowest radially symmetric eigenvalue  $\mu_{\varepsilon,0}$  versus  $\varepsilon$  (solid line), as determined from the numerical solution of (2.2) on  $0 \leq \varepsilon \leq 0.1$ . The limiting value of  $\mu_{\varepsilon,0}$  as  $\varepsilon \rightarrow 0$  is not the lowest eigenvalue  $\mu_{*,0}$  of the unperturbed problem with no hole determined by (2.1).

Using the well-known behavior of the Bessel functions for small argument, we then identify the local behavior as

$$u_0(r) = \alpha_0 r^2 \log r + \mathcal{O}(r^2), \quad \text{as } r \rightarrow 0, \quad \text{where} \quad \alpha_0 \equiv \frac{A\mu_0^2 [J_0(\mu_0) - I_0(\mu_0)]}{2K_0(\mu_0) + \pi Y_0(\mu_0)}. \quad (2.4)$$

It follows that the limiting eigenfunction  $u_0$  is not smooth at the puncture point, but merely differentiable. In terms of this singularity behavior and the constant  $\alpha_0$ , we now derive an expression for the difference  $\mu_0 - \mu_*$  between the lowest eigenvalue of the point constraint problem and that of the unperturbed problem.

We will derive an expression for this difference in a more general context than an annular domain. We consider the limiting point constraint problem, with  $u_0(x_0) = 0$ , given by

$$\Delta^2 u_0 = \lambda_0 u_0, \quad x \in \Omega \setminus \{x_0\}; \quad u_0 = \partial_n u_0 = 0, \quad x \in \partial\Omega; \quad \int_{\Omega} u_0^2 dx = 1; \quad (2.5 a)$$

$$u_0(x) \sim \alpha_0 |x - x_0|^2 \log |x - x_0| + \nabla_x R(x; x_0)|_{x=x_0} \cdot (x - x_0) + \mathcal{O}(|x - x_0|^2), \quad \text{as } x \rightarrow x_0, \quad (2.5 b)$$

and the problem in the absence of a hole, with smooth solution  $u_*$ , satisfying

$$\Delta^2 u_* = \lambda_* u_*, \quad x \in \Omega; \quad u_* = \partial_n u_* = 0, \quad x \in \partial\Omega; \quad \int_{\Omega} u_*^2 dx = 1. \quad (2.6)$$

The following result gives an expression for the difference between any two eigenvalues of (2.5) and (2.6):

**Principal Result 2.1:** *Let  $(u_0, \lambda_0)$  be any simple eigenpair of the limiting problem (2.5) with a point constraint  $u_0(x_0) = 0$  and let  $(u_*, \lambda_*)$  be any simple eigenpair of the problem (2.6) with no hole. Then for  $\langle u_0, u_* \rangle \neq 0$ ,*

$$\lambda_0 - \lambda_* = -\frac{8\pi \alpha_0 u_*(x_0)}{\langle u_0, u_* \rangle}, \quad \text{where} \quad \langle u_0, u_* \rangle \equiv \int_{\Omega} u_0 u_* dx. \quad (2.7)$$

**Proof:** We use Green's second identity over the domain  $\Omega \setminus B(x_0, \sigma)$ , where  $B(x_0, \sigma)$  is a ball of radius  $\sigma \ll 1$  centered at  $x_0$ . This yields that

$$\begin{aligned} (\lambda_0 - \lambda_*) \int_{\Omega \setminus B(x_0, \sigma)} u_0 u_* dx &= \int_{\Omega \setminus B(x_0, \sigma)} (u_* \Delta^2 u_0 - u_0 \Delta^2 u_*) dx \\ &= \int_{\partial B(x_0, \sigma)} (u_* \partial_n (\Delta u_0) - \Delta u_0 \partial_n u_* - u_0 \partial_n (\Delta u_*) + \Delta u_* \partial_n u_0) ds. \end{aligned} \quad (2.8)$$

Then, with  $r = |x - x_0|$  and  $\partial_n = -\partial_r$  on  $B(x_0, \sigma)$ , we use (2.5 b) to calculate as  $r \rightarrow 0$  that

$$u_0 \sim \alpha_0 r^2 \log r + a_c r \cos \theta + a_s r \sin \theta + \dots, \quad u_{0r} \sim 2\alpha_0 r \log r + \alpha_0 r + a_c \cos \theta + a_s \sin \theta + \dots,$$

$$\Delta u_0 \sim 4\alpha_0 [\log r + 1] + \dots, \quad \partial_r \Delta u_0 \sim \frac{4\alpha_0}{r},$$

where  $(x - x_0) = r(\cos \theta, \sin \theta)$  and  $(a_c, a_s) \equiv \nabla_x R(x; x_0)|_{x=x_0}$ . Then, since  $u_*$  is smooth as  $x \rightarrow x_0$ , it follows that only the first term in the boundary integral on  $\partial B(x_0, \sigma)$  in (2.8) is non-vanishing in the limit  $\sigma \rightarrow 0$ . Therefore,

$$(\lambda_0 - \lambda_*) \langle u_0, u_* \rangle = - \lim_{\sigma \rightarrow 0} \int_{\partial B(x_0, \sigma)} u_* \partial_r (\Delta u_0) ds = - \lim_{\sigma \rightarrow 0} 2\pi\sigma \left[ u_*(x_0) \frac{4\alpha_0}{\sigma} \right] = -8\pi \alpha_0 u_*(x_0).$$

In the scenario where  $\langle u_0, u_* \rangle \neq 0$ , we recover the result (2.7). ■

To verify this result in the exactly solvable case of the unit disk with hole centered at the origin, we can without loss of generality set  $A = 1$  for the normalization constants in (2.1) and (2.3 b) and calculate that

$$u_*(0) = 1 - \frac{J_0(\mu_*)}{I_0(\mu_*)}, \quad \alpha_0 = \frac{\mu_0^2 [J_0(\mu_0) - I_0(\mu_0)]}{2K_0(\mu_0) + \pi Y_0(\mu_0)}. \quad (2.9)$$

For the case of the lowest eigenvalue eigenvalues of (2.1) and (2.3 b), we evaluate numerically that

$$\mu_0 = 4.7683, \quad \mu_* = 3.1962, \quad u_*(0) = 1.0557, \quad \alpha_0 = 618.2445, \quad \langle u_0, u_* \rangle = -39.758.$$

This yields  $\lambda_0 - \lambda_* = \mu_0^4 - \mu_*^4 \approx 412.6$  from (2.7), and is in agreement with the numerical results from Fig. 2.

## 2.1 An Exactly Solvable BVP

To motivate the different scalings appearing in §3–§5 below, it is instructive to consider the following radially symmetric model BVP for  $u = u(r)$ :

$$-\delta \Delta^2 u + \Delta u = 1, \quad \varepsilon < r < 1; \quad u(1) = u_r(1) = 0, \quad u(\varepsilon) = u_r(\varepsilon) = 0. \quad (2.10)$$

Here  $\Delta u \equiv u_{rr} + r^{-1}u_r$ , and  $\varepsilon \ll 1$ . The particular solution for (2.10) is  $r^2/4$  while the homogeneous solution is a linear combination of  $\{1, \log r, I_0(r/\sqrt{\delta}), K_0(r/\sqrt{\delta})\}$ . In this way, the exact solution to (2.10) is readily obtained as

$$u_e = \frac{(r^2 - 1)}{4} + \left[ -\frac{c_e}{\sqrt{\delta}} I_0' \left( \frac{1}{\sqrt{\delta}} \right) - \frac{1}{2} - \frac{d_e}{\sqrt{\delta}} K_0' \left( \frac{1}{\sqrt{\delta}} \right) \right] \log r$$

$$+ c_e \left[ I_0 \left( \frac{r}{\sqrt{\delta}} \right) - I_0 \left( \frac{1}{\sqrt{\delta}} \right) \right] + d_e \left[ K_0 \left( \frac{r}{\sqrt{\delta}} \right) - K_0 \left( \frac{1}{\sqrt{\delta}} \right) \right], \quad (2.11 a)$$

where the constants  $c_e$  and  $d_e$  satisfy the  $2 \times 2$  linear system

$$c_e \left[ I_0 \left( \frac{\varepsilon}{\sqrt{\delta}} \right) - \frac{\log \varepsilon}{\sqrt{\delta}} I_0' \left( \frac{1}{\sqrt{\delta}} \right) - I_0 \left( \frac{1}{\sqrt{\delta}} \right) \right] + d_e \left[ K_0 \left( \frac{\varepsilon}{\sqrt{\delta}} \right) - \frac{\log \varepsilon}{\sqrt{\delta}} K_0' \left( \frac{1}{\sqrt{\delta}} \right) - K_0 \left( \frac{1}{\sqrt{\delta}} \right) \right] = \frac{1}{4} - \frac{\varepsilon^2}{4} + \frac{\log \varepsilon}{2}, \quad (2.11 b)$$

$$c_e \left[ I_0' \left( \frac{\varepsilon}{\sqrt{\delta}} \right) - \frac{1}{\varepsilon} I_0' \left( \frac{1}{\sqrt{\delta}} \right) - \sqrt{\delta} I_0 \left( \frac{1}{\sqrt{\delta}} \right) \right] + d_e \left[ K_0' \left( \frac{\varepsilon}{\sqrt{\delta}} \right) - \frac{1}{\varepsilon} K_0' \left( \frac{1}{\sqrt{\delta}} \right) - \sqrt{\delta} K_0 \left( \frac{1}{\sqrt{\delta}} \right) \right] = \frac{\sqrt{\delta}}{2\varepsilon} - \frac{\varepsilon\sqrt{\delta}}{2}. \quad (2.11 c)$$

We now investigate three different asymptotic limits of (2.11).

We first suppose that  $\delta = \mathcal{O}(1)$  and  $\varepsilon \rightarrow 0$ . In (2.11 b) and (2.11 c) we use the small argument expansions  $K_0'(z) \sim -1/z$ ,  $K_0(z) \sim -\log(z/2) - \gamma_e$ , and  $I_0(z) \sim 1$  as  $z \rightarrow 0$  to obtain, after some algebra, that (2.11) reduces to

$$u \sim u_0 \equiv \frac{r^2}{4} + c_0 \left[ I_0 \left( \frac{r}{\sqrt{\delta}} \right) - 1 \right] + d_0 \left[ K_0 \left( \frac{r}{\sqrt{\delta}} \right) + \log r - \log(2\sqrt{\delta}) + \gamma_e \right], \quad (2.12 a)$$

where  $\gamma_e$  is Euler's constant. Here  $c_e \sim c_0$  and  $d_e \sim d_0$  as  $\varepsilon \rightarrow 0$ , where  $c_0$  and  $d_0$  satisfy the linear system

$$c_0 \left[ I_0 \left( \frac{1}{\sqrt{\delta}} \right) - 1 \right] + d_0 \left[ K_0 \left( \frac{1}{\sqrt{\delta}} \right) - \log \left( 2\sqrt{\delta} e^{-\gamma_e} \right) \right] = -\frac{1}{4}, \quad c_0 I_0' \left( \frac{1}{\sqrt{\delta}} \right) + d_0 \left[ K_0' \left( \frac{1}{\sqrt{\delta}} \right) + \sqrt{\delta} \right] = -\frac{\sqrt{\delta}}{2}.$$

The key observation is that the limiting solution (2.12 a) satisfies  $u_0(1) = u_{0r}(1) = 0$  and the point constraint  $u_0(0) = 0$ . We emphasize that this limiting behavior for (2.11) as  $\varepsilon \rightarrow 0$  when  $\delta = \mathcal{O}(1)$ , is not to the smooth unperturbed solution  $u_*$  in the absence of a hole, which is given by

$$u_* = \frac{1}{4} (r^2 - 1) - \frac{\sqrt{\delta}}{2I_0' \left( \frac{1}{\sqrt{\delta}} \right)} \left[ I_0 \left( \frac{r}{\sqrt{\delta}} \right) - I_0 \left( \frac{1}{\sqrt{\delta}} \right) \right]. \quad (2.13)$$

The fact that the limiting solution of (2.10) as  $\varepsilon \rightarrow 0$  with  $\delta = \mathcal{O}(1)$  is not the unperturbed solution  $u_*$  is also apparent from a failure of the method of matched asymptotic expansions. More specifically, if we were to use (2.13) as the leading-order solution in the outer region, then in the inner region, and with local variable  $\rho = r/\varepsilon$ , the leading-order inner problem for (2.10) would be  $\Delta^2 v = 0$  in  $\rho \geq 1$ , subject to  $v = v_\rho = 0$  on  $\rho = 1$  together with the matching condition  $v \rightarrow u_*(0)$  as  $\rho \rightarrow \infty$ . Since  $u_*(0) \neq 0$  from (2.13), and  $v$  must be a linear combination of  $\{1, \log \rho, \rho^2, \rho^2 \log \rho\}$ , it follows that there is no such solution to this inner problem. This, at least formally, suggests that (2.13) cannot be used as the outer solution for (2.10), and instead we must use the solution  $u_0$  in (2.12) satisfying the point constraint  $u_0(0) = 0$  as the outer solution.

The second use of the model problem is to investigate the asymptotic behavior of the solution to (2.10) as  $\delta$  and  $\varepsilon$  both tend to zero. Two cases are considered: Case I:  $\delta = \mathcal{O}(\varepsilon^2)$ . Case II:  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$ . For Case I, we set  $\delta = \delta_0 \varepsilon^2$  in (2.11) and let  $\varepsilon \rightarrow 0$ . Upon using the well-known large argument expansions of  $K_0(z)$  and  $I_0(z)$  as  $z \rightarrow \infty$ , we obtain after some straightforward, but rather lengthy, algebraic manipulations on (2.11) that

$$d_e \sim -\frac{\nu \sqrt{\delta_0}}{4K_0' \left( \frac{1}{\sqrt{\delta_0}} \right)}, \quad \nu \equiv -\frac{1}{\log(\varepsilon e^{-\chi})}, \quad \chi \equiv \frac{\sqrt{\delta_0} K_0 \left( \frac{1}{\sqrt{\delta_0}} \right)}{K_0' \left( \frac{1}{\sqrt{\delta_0}} \right)}; \quad c_e \sim -\frac{\varepsilon \sqrt{\delta_0}}{I_0' \left( \frac{1}{\varepsilon \sqrt{\delta_0}} \right)} \left( \frac{\nu}{4} + \frac{1}{2} \right), \quad (2.14 a)$$

and that the outer limit of (2.11 a) becomes

$$u_e \sim \frac{(r^2 - 1)}{4} - \frac{\nu}{4} \log r, \quad \text{for } \mathcal{O}(\varepsilon) \ll r < 1 - \mathcal{O}(\sqrt{\delta}). \quad (2.14 b)$$

As a remark, if we include a boundary layer correction term in order to satisfy  $u_r = 0$  on  $r = 1$ , a composite expansion for this modified asymptotic solution is readily obtained as

$$u_e \sim \frac{(r^2 - 1)}{4} - \frac{\nu}{4} \log r + \sqrt{\delta} \left( \frac{1}{2} - \frac{\nu}{4} \right) \left( 1 - e^{-(1-r)/\sqrt{\delta}} \right), \quad \text{for } \mathcal{O}(\varepsilon) \ll r \leq 1. \quad (2.14 c)$$

The approximation (2.14 b) can also be obtained by using the method of matched asymptotic expansions directly on the BVP (2.10), with the analysis being very similar to that given, in a more general context, below in §4.2.

Finally, we consider Case II where  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$ . After some algebra, we obtain from (2.11 b) and (2.11 c) that

$$d_e \sim -\frac{\nu_\infty}{4}, \quad \nu_\infty \equiv -\frac{1}{\log \left( 2\sqrt{\delta} e^{-\gamma} \right)}; \quad c_e \sim -\frac{\sqrt{\delta}}{I_0' \left( \frac{1}{\sqrt{\delta}} \right)} \left( \frac{\nu_\infty}{4} - \frac{1}{2} \right), \quad (2.15 a)$$

and that the outer limit of (2.11 a) is

$$u_e \sim \frac{(r^2 - 1)}{4} - \frac{\nu_\infty}{4} \log r, \quad \text{for } \mathcal{O}(\varepsilon) \ll r < 1 - \mathcal{O}(\sqrt{\delta}). \quad (2.15 b)$$

As a remark, if we include a boundary layer correction term in order to satisfy  $u_r = 0$  on  $r = 1$ , a composite expansion for this asymptotic solution is readily obtained as

$$u_e \sim \frac{(r^2 - 1)}{4} - \frac{\nu_\infty}{4} \log r + \sqrt{\delta} \left( \frac{1}{2} - \frac{\nu_\infty}{4} \right) \left( 1 - e^{-(1-r)/\sqrt{\delta}} \right), \quad \text{for } \mathcal{O}(\sqrt{\delta}) \ll r \leq 1. \quad (2.15 c)$$

A key feature of this limiting solution, valid on the range  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$ , is that the outer solution is asymptotically independent of the radius  $\varepsilon$  of the hole. In this sense, this regime exhibits a *transition to the point constraint behavior* associated with the regime  $\delta = \mathcal{O}(1)$ . This limiting solution (2.15 b) can also be re-derived by using the method of matched asymptotic expansions applied to the BVP (2.10), with the analysis being similar to that given below in §4.4. In Fig. 3 we show a favorable comparison between the asymptotic results in (2.14) and (2.15) for Case I and Case II, respectively, and the exact result given in (2.11).

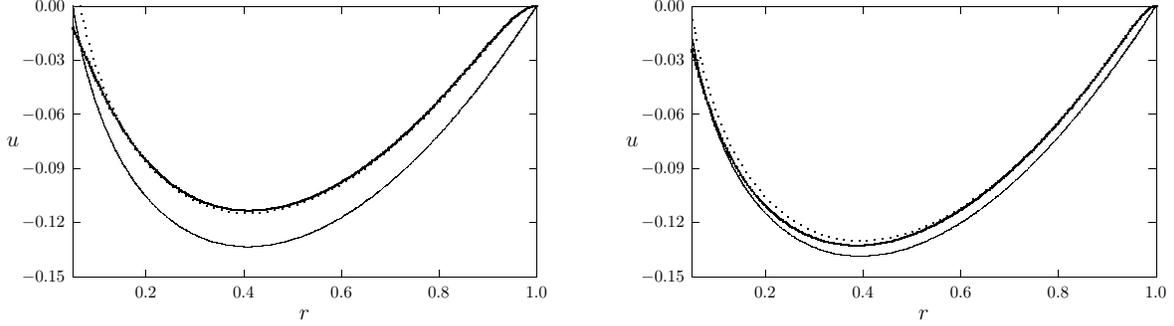


Figure 3. Left panel: The exact solution (2.11) versus  $r$  for  $\delta = 0.002$  and  $\varepsilon = 0.02$  (heavy solid curve) is compared with the asymptotic solution (2.15 b) without boundary layer correction term (solid curve) and with the boundary layer correction term (2.15 c) (dotted curve). Right panel: similar figure for  $\delta = \varepsilon^2$  with  $\varepsilon = 0.02$ , so that  $\delta_0 = 1$ . The asymptotic result is given in (2.14 c) and (2.14 b) with and without the boundary layer correction term, respectively.

### 3 The Mixed-Order Eigenvalue Problem: Asymptotics of the Point Constraint Problem

As motivated by the analysis in §2, in the limit  $\varepsilon \rightarrow 0$  an eigenvalue  $\lambda_\varepsilon$  of (1.3) tends to an eigenvalue  $\lambda_0$  of the point constraint problem

$$-\delta \Delta^2 u_0 + \Delta u_0 + \lambda_0 u_0 = 0, \quad x \in \Omega \setminus \{x_0\}; \quad \int_{\Omega} u_0^2 dx = 1, \quad (3.1 a)$$

$$u_0 = \partial_n u_0 = 0, \quad x \in \partial\Omega; \quad u_0(x_0) = 0. \quad (3.1 b)$$

The effect of this point constraint is that smoothness of  $u_0$  is lost in the sense that  $\Delta u_0 = \mathcal{O}(\log|x - x_0|)$  as  $x \rightarrow x_0$ .

To solve (3.1) it is convenient to introduce the Green's function  $G_\delta(x; x_0, \lambda_0)$  satisfying

$$\Delta^2 G_\delta - \frac{1}{\delta} \Delta G_\delta - \frac{\lambda_0}{\delta} G_\delta = \delta(x - x_0), \quad x \in \Omega; \quad G_\delta = \partial_n G_\delta = 0, \quad x \in \partial\Omega. \quad (3.2 a)$$

Then,  $G_\delta$  can be decomposed in terms of a singular part and a  $C^2$  smooth “regular” part  $R_\delta(x; x_0, \lambda_0)$  as

$$G_\delta(x; x_0, \lambda_0) = \frac{1}{8\pi} |x - x_0|^2 \log|x - x_0| + R_\delta(x; x_0, \lambda_0). \quad (3.2 b)$$

As  $x \rightarrow x_0$ ,  $G_\delta$  in (3.2 b) has the limiting behavior

$$G_\delta(x; x_0, \lambda_0) \sim \frac{1}{8\pi} |x - x_0|^2 \log|x - x_0| + R_\delta(x_0; x_0, \lambda_0) + \nabla_x R_\delta(x; x_0, \lambda_0)|_{x=x_0} + \mathcal{O}(|x - x_0|^2). \quad (3.3)$$

In terms of  $G_\delta$ , the solution to (3.1) is simply  $u_0 = N_0 G_\delta(x; x_0, \lambda_0)$ , where  $\lambda_0$  is a root of

$$R_\delta(x_0; x_0, \lambda_0) = 0, \quad (3.4)$$

and  $N_0 = 1/(\int_{\Omega} G_\delta^2 dx)^{1/2}$ . The roots of the point constraint condition (3.4), which is a transcendental equation in

$\lambda_0$ , give the leading-order eigenvalue asymptotics of (1.3) in that  $\lambda_\varepsilon \rightarrow \lambda_0$  as  $\varepsilon \rightarrow 0$ . We will assume that  $\lambda_0$  is a root of (3.4) of multiplicity one and that  $x_0$  is such that  $u_0$  satisfies the non-degeneracy condition  $\nabla_x R_\delta|_{x=x_0} \neq 0$ .

The goal of this section is to derive an approximation to the point constraint condition (3.4) when  $\delta \ll 1$ . To do so, we will use the method of matched asymptotic expansions to analyze the Green's function of (3.2) as  $\delta \rightarrow 0$ .

In the outer region  $\Omega_{\text{out}}$ , defined as  $\Omega_{\text{out}} = \{x \mid |x - x_0| \gg \mathcal{O}(\sqrt{\delta}) \text{ and } \text{dist}(x, \partial\Omega) \gg \mathcal{O}(\sqrt{\delta})\}$ , we expand  $G_\delta = \mathcal{G}_0 + o(1)$  to obtain from (3.2 a) that

$$\Delta \mathcal{G}_0 + \lambda_0 \mathcal{G}_0 = 0, \quad x \in \Omega_{\text{out}}; \quad \mathcal{G}_0 \rightarrow 0, \quad \text{as } x \rightarrow \partial\Omega. \quad (3.5 a)$$

This effective zero Dirichlet boundary condition for  $\mathcal{G}_0$  as  $x \rightarrow \partial\Omega$  arises as a leading-order condition for matching  $\mathcal{G}_0$  to a boundary-layer solution defined in an  $\mathcal{O}(\sqrt{\delta})$  neighborhood of  $\partial\Omega$ . To construct a solution to (3.2 a) that has a singularity at  $x_0$ , we first impose that  $\mathcal{G}_0$  has a logarithmic singularity as  $x \rightarrow x_0$ , so that

$$\mathcal{G}_0 \sim S \log |x - x_0|, \quad \text{as } x \rightarrow x_0, \quad (3.5 b)$$

for some  $S$  to be determined. The solution to (3.5) is then simply

$$\mathcal{G}_0 = -2\pi S G_h(x; x_0, \lambda_0), \quad (3.6)$$

where  $G_h$  is the Helmholtz Green's function satisfying

$$\Delta G_h + \lambda_0 G_h = -\delta(x - x_0), \quad x \in \Omega; \quad G_h = 0, \quad x \in \partial\Omega, \quad (3.7 a)$$

$$G_h \sim -\frac{1}{2\pi} \log |x - x_0| + R_h(x; x_0, \lambda_0)|_{x=x_0} + \nabla_x R_h(x; x_0, \lambda_0)|_{x=x_0} \cdot (x - x_0) + \cdots, \quad \text{as } x \rightarrow x_0. \quad (3.7 b)$$

Here  $R_{h0} \equiv R_h(x; x_0, \lambda_0)|_{x=x_0}$  and  $\nabla_x R_{h0} \equiv \nabla_x R_h(x; x_0, \lambda_0)|_{x=x_0}$  are the regular part of the Helmholtz Green's function and its gradient, respectively, which depend on  $\lambda_0$  and the domain  $\Omega$ . From (3.6) and (3.7 b), it follows that

$$\mathcal{G}_0 \sim S [\log |x - x_0| - 2\pi R_{h0} - 2\pi \nabla_x R_{h0} \cdot (x - x_0) + \cdots], \quad \text{as } x \rightarrow x_0. \quad (3.8)$$

This behavior will be used as the matching condition for an inner solution in a neighborhood of  $x_0$ .

In the inner region, defined in an  $\mathcal{O}(\sqrt{\delta})$  neighborhood of  $x_0$ , we introduce new variables  $y$  and  $w$  by

$$y = (x - x_0)/\sqrt{\delta}, \quad w(y) = G_\delta(x_0 + \varepsilon y; x_0, \lambda_0). \quad (3.9)$$

We obtain from (3.2 a) that  $w \sim w_0 + \cdots$  where  $w_0$  satisfies

$$\Delta_y^2 w_0 - \Delta_y w_0 = 0, \quad y \in \mathbb{R}^2 \setminus \{0\}. \quad (3.10 a)$$

In terms of the inner variable, the matching condition as  $|y| \rightarrow \infty$ , as obtained from (3.8), is that

$$w_0 \sim S \log |y| + S \log \sqrt{\delta} - 2\pi S R_{h0} - 2\pi S \sqrt{\delta} \nabla_x R_{h0} \cdot y, \quad \text{as } |y| \rightarrow \infty. \quad (3.10 b)$$

Moreover, upon substituting the inner scale  $y = (x - x_0)/\sqrt{\delta}$  into (3.3), we require that

$$w_0 \sim \frac{\delta}{8\pi} |y|^2 \log |y| + a + b \cdot y + \mathcal{O}(|y|^2), \quad \text{as } y \rightarrow 0, \quad (3.11)$$

for some scalar  $a$  and vector  $b$  to be found. We remark that the unknown  $a$  represents the limiting asymptotics of  $R_\delta(x_0; x_0, \lambda_0)$  as  $\delta \rightarrow 0$ , which we seek to determine.

The solution to (3.10) that is bounded as  $y \rightarrow 0$  is simply

$$w_0 = S \left[ \log |y| + \log \left( \sqrt{\delta} \right) - 2\pi R_{h0} + K_0(|y|) \right] - 2\pi S \sqrt{\delta} \nabla_x R_{h0} \cdot y, \quad (3.12)$$

where  $K_0(|y|)$  is the modified Bessel function of the second kind of order zero. To determine  $S$ , we use the well-known refined asymptotics of  $K_0(|y|)$  as  $|y| \rightarrow 0$  from (4.20) below, to obtain in terms of Euler's constant  $\gamma_e \approx 0.5772$  that

$$w_0 \sim -\frac{S}{4} |y|^2 \log |y| + S \left[ \log \left( \sqrt{\delta} \right) - 2\pi R_{h0} + \log 2 - \gamma_e \right] - 2\pi S \sqrt{\delta} \nabla_x R_{h0} \cdot y. \quad (3.13)$$

The final step in the analysis is to choose  $S$  so that the  $\mathcal{O}(|y|^2 \log |y|)$  terms in (3.11) and (3.13) agree, and then identify the constant  $a$  in (3.11) from the  $\mathcal{O}(1)$  term in (3.13). In this way, we obtain that  $S = -\delta/(2\pi)$  and that

$$R_\delta(x_0; x_0, \lambda_0) \sim -\frac{\delta}{2\pi} \left[ \log \left( 2\sqrt{\delta} e^{-\gamma\epsilon} \right) - 2\pi R_{h0} \right], \quad \text{as } \delta \rightarrow 0. \quad (3.14)$$

Finally, upon substituting (3.14) into the point constraint condition (3.4), it follows for  $\delta \rightarrow 0$  that  $\lambda_0$  is a root of

$$R_{h0} = -\frac{1}{2\pi\nu_\infty}, \quad \text{where } \nu_\infty \equiv -1/\log \left( 2\sqrt{\delta} e^{-\gamma\epsilon} \right). \quad (3.15)$$

Here  $R_{h0} \equiv R_h(x_0; x_0, \lambda_0)$  is the regular part of the Helmholtz Green's function as defined by (3.7).

This analysis of the limiting behavior as  $\delta \rightarrow 0$  of the point constraint condition (3.4) is insufficient for identifying the range of  $\delta \ll 1$  and  $\epsilon \ll 1$  for which (3.15) holds as an approximation to an eigenvalue of (1.3). In the more refined analysis of §4.4, leading to Principal Result 4.4, it is shown that (3.15) yields the leading-order asymptotics for an eigenvalue of (1.3) on the range  $\mathcal{O}(\epsilon^2) \ll \delta \ll \mathcal{O}(1)$  for a hole  $\Omega_\epsilon$  of arbitrary shape centered at  $x_0$ .

#### 4 Asymptotics for $\delta \rightarrow 0$ of the Mixed-Order Eigenvalue Problem

In this section we investigate the limiting behavior as  $\delta \rightarrow 0$  of the mixed-order problem

$$-\delta\Delta^2 u + \Delta u + \lambda u = 0, \quad x \in \Omega \setminus \Omega_\epsilon; \quad \int_{\Omega \setminus \Omega_\epsilon} u^2 dx = 1, \quad (4.1 a)$$

$$u = \partial_n u = 0, \quad x \in \partial\Omega; \quad u = \partial_n u = 0, \quad x \in \partial\Omega_\epsilon. \quad (4.1 b)$$

Here  $\Omega_\epsilon$  is a hole of diameter  $\mathcal{O}(\epsilon)$  that shrinks uniformly to a point  $x = x_0 \in \Omega$  as  $\epsilon \rightarrow 0$ . In our analysis of (4.1) in §4.2–4.4 we will consider the limit  $\epsilon \rightarrow 0$  for the three asymptotic ranges of  $\delta \ll 1$  given by  $\delta = \mathcal{O}(\epsilon^2)$ ,  $\delta \ll \mathcal{O}(\epsilon^2)$ , and  $\mathcal{O}(\epsilon^2) \ll \delta \ll 1$ . For each of these regimes, we will determine asymptotic approximations for any  $\mathcal{O}(1)$  simple eigenvalues of (4.1). However, first, in order to isolate the effect of the outer boundary  $\partial\Omega$ , in §4.1 we begin by calculating the eigenvalues and eigenfunctions of (4.1) for the case  $\delta \ll 1$  when there is no hole.

##### 4.1 Boundary layer at $\partial\Omega$

In this subsection we analyze the perturbation of the eigenvalues  $\lambda$  due to the fourth order term  $-\delta\Delta^2 u$  with  $\delta \ll 1$  in the absence of a hole. Assuming, for simplicity, that the boundary  $\partial\Omega$  is  $C^2$ , this eigenproblem is

$$-\delta\Delta^2 u + \Delta u + \lambda u = 0, \quad x \in \Omega; \quad u = \partial_n u = 0, \quad x \in \partial\Omega; \quad \int_{\Omega} u^2 dx = 1. \quad (4.2)$$

In the limit  $\delta \rightarrow 0$ , the formal leading order problem for (4.2) is  $\Delta u + \lambda u = 0$ . For this reduced problem, both boundary conditions for  $u$  cannot be applied on  $\partial\Omega$ . To understand how the full boundary conditions are enforced, a boundary layer is introduced in the vicinity of  $\partial\Omega$ . Since  $\partial\Omega$  is arbitrary, but smooth, it is convenient to employ an orthogonal system  $(\eta, s)$ , where  $\eta > 0$  measures the perpendicular distance from  $x \in \Omega$  to  $\partial\Omega$ , while on  $\partial\Omega$ ,  $s$  measures arc-length. In terms of these coordinates, valid in a neighborhood of  $\partial\Omega$ , (4.2) becomes

$$-\delta \left[ \partial_{\eta\eta} - \frac{\kappa}{1-\kappa\eta} \partial_\eta + \frac{1}{1-\kappa\eta} \partial_s \left( \frac{1}{1-\kappa\eta} \partial_s \right) \right]^2 u + \left[ \partial_{\eta\eta} - \frac{\kappa}{1-\kappa\eta} \partial_\eta + \frac{1}{1-\kappa\eta} \partial_s \left( \frac{1}{1-\kappa\eta} \partial_s \right) \right] u + \lambda u = 0. \quad (4.3)$$

Here  $\kappa = \kappa(s)$  is the curvature of  $\partial\Omega$ , with  $\kappa = 1$  for the unit disk. Equation (4.3) is valid when  $\eta < 1/(\max_{\partial\Omega} \kappa)$ .

In the inner boundary layer region near  $\partial\Omega$  of extent  $\mathcal{O}(\delta^{1/2})$ , we introduce the inner variables and inner expansion

$$\hat{\eta} = \eta/\delta^{1/2}, \quad u = \delta^{1/2} v, \quad v = v_0 + \delta^{1/2} v_1 + \delta v_2 + \dots. \quad (4.4)$$

Upon substituting (4.4) into (4.3) and collecting powers of  $\delta$ , we obtain the following sequence of problems on  $\hat{\eta} \geq 0$ :

$$-v_0 \hat{\eta} \hat{\eta} \hat{\eta} + v_0 \hat{\eta} = 0, \quad v_0 = v_0 \hat{\eta} = 0, \quad \text{on } \hat{\eta} = 0, \quad (4.5 a)$$

$$-v_1 \hat{\eta} \hat{\eta} \hat{\eta} + v_1 \hat{\eta} = -2\kappa v_0 \hat{\eta} \hat{\eta} + \kappa v_0 \hat{\eta}, \quad v_1 = v_1 \hat{\eta} = 0, \quad \text{on } \hat{\eta} = 0. \quad (4.5 b)$$

To asymptotically match the inner solution to an outer solution, as constructed below, we require that the solution to (4.5) has no exponential growth as  $\hat{\eta} \rightarrow +\infty$ . In this way, the solution to (4.5) is

$$v_0 = -c_0 + c_0 \hat{\eta} + c_0 e^{-\hat{\eta}}; \quad v_1 = -c_1 + \left(c_1 - \frac{\kappa c_0}{2}\right) \hat{\eta} + c_1 e^{-\hat{\eta}} + \frac{\kappa c_0}{2} \hat{\eta}^2 + \frac{\kappa c_0}{2} \hat{\eta} e^{-\hat{\eta}}, \quad (4.6)$$

where  $c_0(s)$  and  $c_1(s)$  are to be determined. As  $\hat{\eta} \rightarrow \infty$ , these solutions have the dominant asymptotic behavior

$$v_0 \sim -c_0 + c_0 \hat{\eta}, \quad v_1 \sim -c_1 + \left(c_1 - \frac{\kappa c_0}{2}\right) \hat{\eta} + \frac{\kappa c_0}{2} \hat{\eta}^2.$$

In terms of the outer variable  $\eta = \delta^{1/2} \hat{\eta}$ , the far-field behavior of the two-term inner expansion is

$$u \sim c_0 \eta + \frac{\kappa c_0}{2} \eta^2 + \delta^{1/2} \left[-c_0 + \eta \left(c_1 - \frac{\kappa c_0}{2}\right)\right] + \delta [-c_1 + \mathcal{O}(\eta)] + \dots \quad (4.7)$$

In the outer region, defined away from an  $\mathcal{O}(\delta^{1/2})$  distance from  $\partial\Omega$ , we pose the outer expansion

$$u = u_0 + \delta^{1/2} u_1 + \delta u_2 + \dots \quad (4.8)$$

In terms of the boundary-fitted orthogonal coordinate system, the local behavior of this outer solution as  $\eta \rightarrow 0$  is

$$u \sim u_0 + \eta \partial_\eta u_0 + \frac{\eta^2}{2} \partial_{\eta\eta} u_0 + \delta^{1/2} [u_1 + \eta u_{1\eta} + \dots] + \delta [u_2 + \dots] + \dots, \quad (4.9)$$

where  $u_0$ ,  $u_1$  and their derivatives are to be evaluated on  $\eta = 0$ . Upon comparing (4.9) with the required matching condition (4.7), and noting that the outer normal derivative  $\partial_n u$  on  $\partial\Omega$  is simply  $\partial_n u = -\partial_\eta u$ , we obtain

$$u_0 = 0, \quad u_1 = \partial_n u_0, \quad u_2 = \frac{\kappa}{2} \partial_n u_0 + \partial_n u_1, \quad x \in \partial\Omega. \quad (4.10)$$

Moreover, from this matching condition,  $c_0$  and  $c_1$  in (4.6) are given by

$$c_0 = -\partial_n u_0, \quad c_1 = -\frac{\kappa}{2} \partial_n u_0 - \partial_n u_1, \quad x \in \partial\Omega. \quad (4.11)$$

Next, we substitute (4.8), together with the eigenvalue expansion  $\lambda = \lambda_0 + \delta^{1/2} \lambda_1 + \delta \lambda_2 + \dots$ , into (4.2), and collect powers of  $\delta$ , to obtain a sequence of outer problems. The effective boundary conditions as  $x \rightarrow \partial\Omega$  for these outer problems are given by (4.10). In this way, we obtain that

$$\Delta u_0 + \lambda_0 u_0 = 0, \quad x \in \Omega; \quad u_0 = 0, \quad x \in \partial\Omega; \quad \int_\Omega u_0^2 dx = 1, \quad (4.12 a)$$

$$\Delta u_1 + \lambda_0 u_1 = -\lambda_1 u_0, \quad x \in \Omega; \quad u_1 = \partial_n u_0, \quad x \in \partial\Omega; \quad \int_\Omega u_0 u_1 dx = 0, \quad (4.12 b)$$

$$\Delta u_2 + \lambda_0 u_2 = (\lambda_0^2 - \lambda_2) u_0 - \lambda_1 u_1, \quad x \in \Omega; \quad u_2 = \partial_n u_1 + \frac{\kappa}{2} \partial_n u_0, \quad x \in \partial\Omega; \quad \int_\Omega (2u_0 u_2 + u_1^2) dx = 0. \quad (4.12 c)$$

We assume that  $(u_0, \lambda_0)$  is a base eigenpair of (4.12 a) of multiplicity one. Solvability conditions are then applied to (4.12 b) and (4.12 c) to fix the values of  $\lambda_1$  and  $\lambda_2$ , and the solutions  $u_1$  and  $u_2$  are made unique from the integral constraints in (4.12 b) and (4.12 c). In terms of the unique  $u_0$  and  $u_1$ , the functions  $c_0(s)$  and  $c_1(s)$ , as needed in the inner solution (4.6), are calculated from (4.11). Our result is summarized in the following formal statement.

**Principal Result 4.1:** *Let  $(u_0, \lambda_0)$  be an eigenpair of (4.12 a) with multiplicity one, and assume  $\partial\Omega$  is smooth. Then, for  $\delta \ll 1$ , and with  $\kappa(s)$  denoting the curvature of  $\partial\Omega$ , there is an eigenvalue of (4.2) with asymptotics*

$$\lambda(\delta) = \lambda_0 + \delta^{1/2} \left[ \int_{\partial\Omega} (\partial_n u_0)^2 ds \right] + \delta \left[ \lambda_0^2 + \int_{\partial\Omega} \partial_n u_0 (\partial_n u_1 + \frac{\kappa}{2} \partial_n u_0) ds \right] + \mathcal{O}(\delta^{3/2}). \quad (4.13)$$

We conclude that the biharmonic term and the extra boundary condition  $\partial_n u = 0$  on  $\partial\Omega$  induce an  $\mathcal{O}(\sqrt{\delta})$  correction to the eigenvalue of the Laplacian. In the three subsections below, the various scalings  $\delta = \delta(\varepsilon)$  that are investigated as  $\varepsilon \rightarrow 0$  in (4.1) are all such that eigenvalue correction terms due to the boundary layer are asymptotically smaller than those generated by the hole. However, by including a transcendentally small boundary layer contribution to the eigenvalue approximation a quantitatively more accurate result is obtained at moderately small values of  $\delta$ .

#### 4.2 The Distinguished Limit $\delta = \delta_0 \varepsilon^2$

In the limit  $\varepsilon \rightarrow 0$ , in this subsection we study the eigenvalue problem

$$-\delta_0 \varepsilon^2 \Delta^2 u + \Delta u + \lambda u = 0, \quad x \in \Omega; \quad \int_{\Omega \setminus \Omega_\varepsilon} u^2 dx = 1, \quad (4.14 a)$$

$$u = \partial_n u = 0, \quad x \in \partial\Omega; \quad u = \partial_n u = 0, \quad x \in \partial\Omega_\varepsilon, \quad (4.14 b)$$

where  $\delta_0 = \mathcal{O}(1)$  is a positive parameter controlling the influence of the bi-Laplacian term in (4.14).

Our analysis begins by introducing a canonical local problem defined near the hole  $\Omega_\varepsilon$ . In terms of the local variables

$$y = \varepsilon^{-1}(x - x_0), \quad v(y) = u(x_0 + \varepsilon y), \quad (4.15)$$

we obtain the following inner problem from (4.14):

$$-\delta_0 \Delta_y^2 v + \Delta_y v + \varepsilon^2 \lambda v = 0, \quad y \in \mathbb{R}^2 \setminus \Omega_0; \quad v = \partial_n v = 0, \quad x \in \partial\Omega_0, \quad (4.16)$$

where  $\Omega_\varepsilon = x_0 + \varepsilon\Omega_0$  and  $\Omega_0$  is the magnified domain of the hole. We define the *canonical* solution  $v_c$  as the solution to (4.16), upon neglecting the  $\mathcal{O}(\varepsilon^2)$  term, that satisfies  $v_c \sim \log|y|$  as  $|y| \rightarrow \infty$ . As such,  $v_c$  is taken to satisfy

$$-\delta_0 \Delta_y^2 v_c + \Delta_y v_c = 0, \quad y \in \mathbb{R}^2 \setminus \Omega_0; \quad v_c = \partial_n v_c = 0, \quad y \in \partial\Omega_0; \quad (4.17 a)$$

$$v_c \sim \log|y| + \chi(\delta_0) + \mathcal{O}(1), \quad \text{as } |y| \rightarrow \infty. \quad (4.17 b)$$

Here the  $\mathcal{O}(1)$  constant  $\chi(\delta_0)$  in the far-field behavior of  $v_c$  depends on both  $\delta_0$  and the shape of the hole  $\Omega_0$ . For an arbitrary shaped hole  $\Omega_0$ ,  $\chi(\delta_0)$  must be calculated numerically.

For the special case where  $\Omega_\varepsilon$  is the circular disk  $|x - x_0| \leq \varepsilon$ , so that  $\Omega_0$  is the unit disk  $|y| \leq 1$ , we can find  $\chi(\delta_0)$  analytically. To do so, we note that any radially symmetric solution of  $-\delta_0 \Delta_y^2 v_c + \Delta_y v_c = 0$  is a linear combination of  $\{1, \log \rho, K_0(\rho/\sqrt{\delta_0}), I_0(\rho/\sqrt{\delta_0})\}$ . Here  $\rho = |y|$  while  $I_0(z)$  and  $K_0(z)$  denote modified Bessel functions of the first and second kinds of order zero. To ensure that  $v_c$  grows only logarithmically as  $\rho \rightarrow \infty$  we must eliminate the  $I_0$  component. A simple calculation yields that

$$v_c = \log \rho + a + bK_0\left(\rho/\sqrt{\delta}\right); \quad a = -\frac{\sqrt{\delta_0}K_0(1/\sqrt{\delta_0})}{K_1(1/\sqrt{\delta_0})}, \quad b = \frac{\sqrt{\delta_0}}{K_1(1/\sqrt{\delta_0})}. \quad (4.18)$$

Then, upon comparing (4.18) with (4.17 b), and using  $K_0(z) \rightarrow 0$  as  $z \rightarrow +\infty$ , we readily identify  $\chi(\delta_0)$  in (4.17 b) as

$$\chi(\delta_0) = -\frac{\sqrt{\delta_0}K_0(1/\sqrt{\delta_0})}{K_1(1/\sqrt{\delta_0})}. \quad (4.19)$$

To determine the asymptotics of  $\chi(\delta_0)$  as  $\delta_0 \rightarrow \infty$  we use the well-known asymptotics for  $z \rightarrow 0$  given by

$$K_0(z) \sim -\log z + \log 2 - \gamma_e - \frac{1}{4}z^2 \log z + \frac{z^2}{4}(\log 2 + 1 - \gamma_e), \quad K_1(z) \sim \frac{1}{z} + \frac{z}{4}(2\gamma_e - 1) + \frac{z}{2} \log\left(\frac{z}{2}\right), \quad (4.20)$$

where  $\gamma_e \approx 0.5772$  is Euler's constant. In contrast, the asymptotics of  $\chi(\delta_0)$  for  $\delta_0 \rightarrow 0$  relies on the identity

$K_0(z)/K_1(z) \sim 1 - 1/(2z) + 51/(128z^2)$  as  $z \rightarrow \infty$ . In this way, we readily calculate that

$$\chi \sim -\log\left(2\sqrt{\delta_0}\right) + \gamma_e - \frac{1}{2\delta_0} \left[\log\left(2\sqrt{\delta_0}\right)\right]^2 + \frac{1}{\delta_0} \left(\gamma_e - \frac{1}{2}\right) \log\left(2\sqrt{\delta_0}\right) - \frac{(1 - 2\gamma_e + 2\gamma_e^2)}{4\delta_0}, \quad \text{as } \delta_0 \rightarrow \infty, \quad (4.21 a)$$

$$\chi \sim -\sqrt{\delta_0} + \frac{\delta_0}{2} - \frac{51}{128}\delta_0^{3/2} + \mathcal{O}(\delta_0^2), \quad \text{as } \delta_0 \rightarrow 0. \quad (4.21 b)$$

In Fig. 4 we plot  $\chi(\delta_0)$ , as computed from (4.19), and compare it with the limiting asymptotic behavior in (4.21).

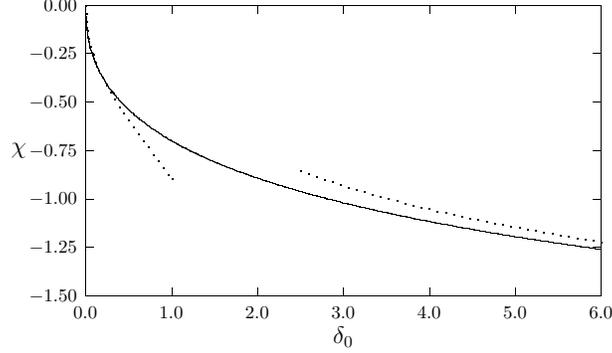


Figure 4. Plot of  $\chi(\delta_0)$  versus  $\delta_0$  for a circular hole of radius  $\varepsilon$  as computed numerically from (4.19) (solid curve). The dotted curves are the approximations in (4.21) that are valid for either  $\delta_0 \ll 1$  or for  $\delta_0 \gg 1$ .

Finally, it is convenient to introduce the constant  $d = d(\delta_0) \equiv e^{-\chi(\delta_0)}$  defined so that

$$v_c \sim \log|y| - \log d + \mathcal{O}(1), \quad \text{as } |y| \rightarrow \infty; \quad d \equiv e^{-\chi(\delta_0)}. \quad (4.22)$$

When  $\Omega_\varepsilon$  is the disk of radius  $\varepsilon$ ,  $d$  can be calculated from  $\chi(\delta_0)$  in (4.19). From (4.21), it has the leading-order behavior

$$d \sim 2\sqrt{\delta_0} e^{-\gamma_e}, \quad \text{as } \delta_0 \rightarrow \infty; \quad d \sim 1 + \sqrt{\delta_0} + \mathcal{O}(\delta_0^{3/2}), \quad \text{as } \delta_0 \rightarrow 0. \quad (4.23)$$

Next, we construct an infinite asymptotic series in powers of  $\nu \equiv -1/\log(\varepsilon d)$  for a simple eigenvalue of (4.1). In terms of the canonical inner solution  $v_c$  and unknown constants  $c_j$ , we expand the inner solution for (4.16) as

$$v \sim \nu \sum_{j=0}^{\infty} \nu^j c_j v_c, \quad \nu \equiv -1/\log(\varepsilon d), \quad d \equiv e^{-\chi(\delta_0)}. \quad (4.24)$$

To determine the far-field behavior of (4.24), we use (4.22) in (4.24) and write the resulting expression in terms of the outer variable. This yields the following matching condition for the outer solution of (4.1):

$$u \sim c_0 + \sum_{j=1}^{\infty} \nu^j [c_{j-1} \log|x - x_0| + c_j], \quad \text{as } x \rightarrow x_0. \quad (4.25)$$

In the outer region, where  $|x - x_0| \gg \mathcal{O}(\varepsilon)$  and  $\text{dist}(x, \partial\Omega) \gg \mathcal{O}(\delta^{1/2}) = \mathcal{O}(\varepsilon)$ , we expand the eigenvalue  $\lambda$  and the outer eigenfunction  $u$  as

$$u \sim u_0 + \sum_{j=1}^{\infty} \nu^j u_j + \dots, \quad \lambda \sim \lambda_0 + \sum_{j=1}^{\infty} \nu^j \lambda_j + \dots. \quad (4.26)$$

Here  $u_0(x)$  and  $\lambda_0$  is assumed to be any simple eigenpair of the unperturbed problem (4.12 a) for which  $u_0(x_0) \neq 0$ . The leading-order matching condition from (4.25) is that  $c_0 = u_0(x_0)$ . Since  $\nu^j \gg \mathcal{O}(\varepsilon)$  for any  $j \geq 0$ , the correction to  $\lambda_0$  induced by the boundary layer near  $\partial\Omega$  is asymptotically smaller than any term in the infinite series (4.26). As such, in our analysis of the outer region, we can neglect both the boundary layer near  $\partial\Omega$  and the term  $-\delta\varepsilon^2\Delta^2$  in

(4.1), while imposing  $u \rightarrow 0$  as  $x \rightarrow \partial\Omega$ . Thus, upon substituting (4.26) into (4.1) and the matching condition (4.25), we obtain upon equating powers of  $\nu$  that  $u_j$  for  $j \geq 1$  satisfies

$$\Delta u_j + \lambda_0 u_j = -\lambda_j u_0 - (1 - \delta_{1j}) \sum_{i=1}^{j-1} \lambda_i u_{j-i} u_0, \quad x \in \Omega \setminus \{x_0\}; \quad u_j = 0, \quad x \in \partial\Omega, \quad (4.27 a)$$

$$u_j \sim c_{j-1} \log|x - x_0| + c_j, \quad \text{as } x \rightarrow x_0, \quad (4.27 b)$$

$$\int_{\Omega} u_0 u_j dx = -\frac{1}{2}(1 - \delta_{1j}) \sum_{i=1}^{j-1} \int_{\Omega} u_i u_{j-i} dx, \quad (4.27 c)$$

where  $c_0 = u_0(x_0) \neq 0$  and  $\delta_{1j}$  is the usual Kronecker symbol.

The coefficients  $c_j$  for  $j \geq 1$  and the eigenvalue corrections  $\lambda_j$  for  $j \geq 1$  are determined recursively from (4.27). Suppose that  $c_{j-1}$  and  $u_i$  for  $0 \leq i \leq j-1$  are known. Then, the solvability condition for (4.27) yields that

$$\lambda_j = 2\pi c_{j-1} u_0(x_0) - (1 - \delta_{1j}) \sum_{i=1}^{j-1} \lambda_i \int_{\Omega} u_{j-i} u_0 dx, \quad \text{for } j \geq 1. \quad (4.28)$$

With  $\lambda_j$  now determined, we can solve (4.27 a) with  $u_j \sim c_{j-1} \log|x - x_0|$  as  $x \rightarrow x_0$  to obtain  $u_j$  to within an additive multiple of  $u_0$ . This multiple is then determined uniquely by the normalization condition (4.27 c). Finally, with  $u_j$  now uniquely determined, we calculate the constant  $c_j$  from the limiting behavior  $c_j = \lim_{x \rightarrow x_0} (u_j - c_{j-1} \log|x - x_0|)$ . This process is repeated recursively in  $j$ . It is initialized with  $c_0 = u_0(x_0)$ , where  $(u_0, \lambda_0)$  is an eigenpair of (4.12 a).

We illustrate this procedure by deriving a three-term expansion for  $\lambda$ . For  $j = 1$ , (4.28) yields that  $\lambda_1 = 2\pi [u_0(x_0)]^2$ . To identify  $u_1$  from (4.27) it is convenient to introduce the uniquely-defined  $G_m(x; x_0)$  by

$$\Delta G_m + \lambda_0 G_m = u_0(x_0) u_0(x) - \delta(x - x_0), \quad x \in \Omega; \quad G_m = 0, \quad x \in \partial\Omega \quad (4.29 a)$$

$$G_m(x; x_0) \sim -\frac{1}{2\pi} \log|x - x_0| + R_m(x_0) + o(1), \quad \text{as } x \rightarrow x_0; \quad \int_{\Omega} G_m u_0 dx = 0, \quad (4.29 b)$$

where  $R_m(x_0)$  is the regular part of the modified Green's function  $G_m$ . The solution to (4.27) with  $j = 1$  is simply

$$u_1 = -2\pi u_0(x_0) G_m(x; x_0). \quad (4.30)$$

Next, expanding  $u_1$  as  $x \rightarrow x_0$  and comparing with (4.27 b) for  $j = 1$ , we get  $c_1 = -2\pi u_0(x_0) R_m(x_0)$ . Finally, since  $\int_{\Omega} u_0 u_1 dx = 0$ , (4.28) for  $j = 2$  yields that  $\lambda_2 = -4\pi^2 [u_0(x_0)]^2 R_m(x_0)$ . We summarize the result as follows:

**Principal Result 4.1:** *Let  $(u_0, \lambda_0)$  be an simple eigenpair of (4.12 a) with  $u_0(x_0) \neq 0$ . Then, for  $\delta = \varepsilon^2 \delta_0$  with  $\delta_0 = \mathcal{O}(1)$ , (4.14) has an eigenvalue with three-term asymptotics*

$$\lambda \sim \lambda_0 + 2\pi [u_0(x_0)]^2 \nu - 4\pi^2 [u_0(x_0)]^2 R_m(x_0) \nu^2 + \mathcal{O}(\nu^3); \quad \nu \equiv -1/\log(\varepsilon d), \quad d = e^{-\chi(\delta_0)}, \quad (4.31)$$

where  $\chi(\delta_0)$  is defined in (4.17 b) in terms of the far-field behavior of the canonical inner solution  $v_c$ . For a circular hole of radius  $\varepsilon$ ,  $\chi(\delta_0)$  is given in (4.19). Here  $R_m(x_0)$  is the regular part of the modified Green's function in (4.29).

We conclude this section by deriving a transcendental equation for  $\lambda$  that has the effect of summing the infinite logarithmic series in (4.26). Since the analysis is similar to that in [23], we only briefly outline it here.

In the inner region, the solution to (4.16) is given asymptotically in terms of some unknown function  $\mathcal{C}(\nu)$  as

$$v \sim \mathcal{C}(\nu) \nu v_c. \quad (4.32)$$

Upon using  $v_c \sim \log|y| - \log d + o(1)$  as  $|y| \rightarrow \infty$  in (4.32), we derive the matching condition for the outer solution:

$$u \sim \mathcal{C} \nu \log|x - x_0| + \mathcal{C}, \quad \text{as } x \rightarrow x_0. \quad (4.33)$$

In the outer region, instead of expanding  $u$  and  $\lambda$  term-by-term in powers of  $\nu$  as in (4.26), we expand  $u = u^*(x; \nu) + \text{t.s.t.}$  and  $\lambda = \lambda^*(\nu) + \text{t.s.t.}$ , where t.s.t. indicates terms that are transcendentally small in comparison with  $\nu$ . In addition,

$u^*$  is to satisfy the matching condition (4.33). In this way, we find that  $u^*$  and  $\lambda^*$  satisfy

$$\Delta u^* + \lambda^* u^* = 0, \quad x \in \Omega \setminus \{x_0\}; \quad u^* = 0, \quad x \in \partial\Omega; \quad \int_{\Omega} [u^*]^2 dx = 1. \quad (4.34 a)$$

$$u^* \sim \mathcal{C}\nu \log|x - x_0| + \mathcal{C}, \quad \text{as } x \rightarrow x_0. \quad (4.34 b)$$

The singularity structure (4.34 b) specifies both the regular and singular part of the singularity. As such, it provides a constraint to determine  $\lambda^*$ . The solution to (4.34) is

$$u^* = -2\pi\mathcal{C}(\nu)G_h(x; x_0, \lambda^*), \quad (4.35)$$

where  $\mathcal{C}$  is found from the condition  $\int_{\Omega} [u^*]^2 dx = 1$ . Here  $G_h$  is the Helmholtz Green's function satisfying

$$\Delta G_h + \lambda^* G_h = -\delta(x - x_0), \quad x \in \Omega; \quad G_h = 0, \quad x \in \partial\Omega, \quad (4.36 a)$$

$$G_h \sim -\frac{1}{2\pi} \log|x - x_0| + R_h(x_0; \lambda^*) + o(1), \quad \text{as } x \rightarrow x_0, \quad (4.36 b)$$

where  $R_h(x_0; \lambda^*)$  is the regular part of  $G_h$ . Then, by expanding  $u^*$  in (4.35) as  $x \rightarrow x_0$  and comparing the resulting expression with (4.34 b), we get  $-2\pi\nu R_h \mathcal{C} = \mathcal{C}$ , which is an equation for  $\lambda^*$ . We summarize the result as follows:

**Principal Result 4.2:** For  $\delta = \varepsilon^2 \delta_0$  with  $\delta_0 = \mathcal{O}(1)$ , there is an eigenvalue  $\lambda$  of (4.14) with  $\lambda \sim \lambda^*(\nu) + t.s.t.$ , where  $\lambda^*(\nu)$  satisfies the transcendental equation

$$R_h(x_0; \lambda^*) = -\frac{1}{2\pi\nu}, \quad \nu \equiv -1/\log(\varepsilon d), \quad d = e^{-\chi(\delta_0)}, \quad (4.37)$$

where  $\chi(\delta_0)$  is defined by (4.17 b), and is given in (4.19) for a circular hole of radius  $\varepsilon$ . Here  $R_h$  is the regular part of the Helmholtz Green's function defined in (4.36). Since  $R_h$  is unbounded as  $\lambda^* \rightarrow \lambda_0$  it follows from (4.37) that  $\lambda^* \rightarrow \lambda_0$  as  $\nu \rightarrow 0$ , where  $\lambda_0$  is an eigenvalue of (4.12 a).

As a test of Principal Result 4.2, we consider the exactly solvable case of the annulus  $\varepsilon < r < 1$ . By factorizing the operator  $-\delta\Delta^2 + \Delta - \lambda$  as a quadratic in  $\Delta$ , we find that the radially symmetric solutions of (4.2) are spanned by  $\{J_0(\xi_1 r), Y_0(\xi_1 r), K_0(\xi_2 r), I_0(\xi_2 r)\}$  where

$$\xi_1 \equiv \sqrt{\frac{-1 + \sqrt{1 + 4\delta\lambda}}{2\delta}}, \quad \xi_2 \equiv \sqrt{\frac{1 + \sqrt{1 + 4\delta\lambda}}{2\delta}}. \quad (4.38)$$

With the clamped boundary conditions  $u_0 = \partial_r u_0 = 0$  on  $r = 1$  and  $r = \varepsilon$ , the eigenvalues are determined

$$\det \begin{bmatrix} J_0(\xi_1) & Y_0(\xi_1) & K_0(\xi_2) & I_0(\xi_2) \\ \xi_1 J_1(\xi_1) & \xi_1 Y_1(\xi_1) & \xi_2 K_1(\xi_2) & -\xi_2 I_1(\xi_2) \\ J_0(\xi_1 \varepsilon) & Y_0(\xi_1 \varepsilon) & K_0(\xi_2 \varepsilon) & I_0(\xi_2 \varepsilon) \\ \xi_1 J_1(\xi_1 \varepsilon) & \xi_1 Y_1(\xi_1 \varepsilon) & \xi_2 K_1(\xi_2 \varepsilon) & -\xi_2 I_1(\xi_2 \varepsilon) \end{bmatrix} = 0. \quad (4.39)$$

To test the accuracy of Principal Result 4.2, we set  $\delta = \varepsilon^2$  ( $\delta_0 = 1$ ) in (4.38) and numerically obtain the zeros of (4.39) over a range of  $\varepsilon$  values. For the particular case  $x_0 = 0$ , the Helmholtz Green's function satisfying (4.36) is a linear combination  $J_0(\sqrt{\lambda^*} r)$  and  $Y_0(\sqrt{\lambda^*} r)$ . The regular part  $R_h(0, \lambda^*)$  is readily calculated to be

$$R_h(0; \lambda^*) = -\frac{1}{2\pi} \left[ -\log 2 + \gamma_e + \log(\sqrt{\lambda^*}) \right] + \frac{1}{4} \frac{Y_0(\sqrt{\lambda^*})}{J_0(\sqrt{\lambda^*})}, \quad (4.40)$$

which allows for a numerical solution of the transcendental equation (4.37).

In Fig. 5 a comparison of the lowest eigenvalue from the asymptotic prediction (4.37) and the exact solution shows agreement between the two theories as  $\varepsilon \rightarrow 0$ . In particular the asymptotic result shows the limiting behavior  $\lambda^* \rightarrow \lambda_0$  as  $\varepsilon \rightarrow 0$  where  $\lambda_0$  is determined by (4.12 a). For this particular example,  $\lambda_0 = z_0^2 \approx 5.7832$  where  $J_0(z_0) = 0$ .

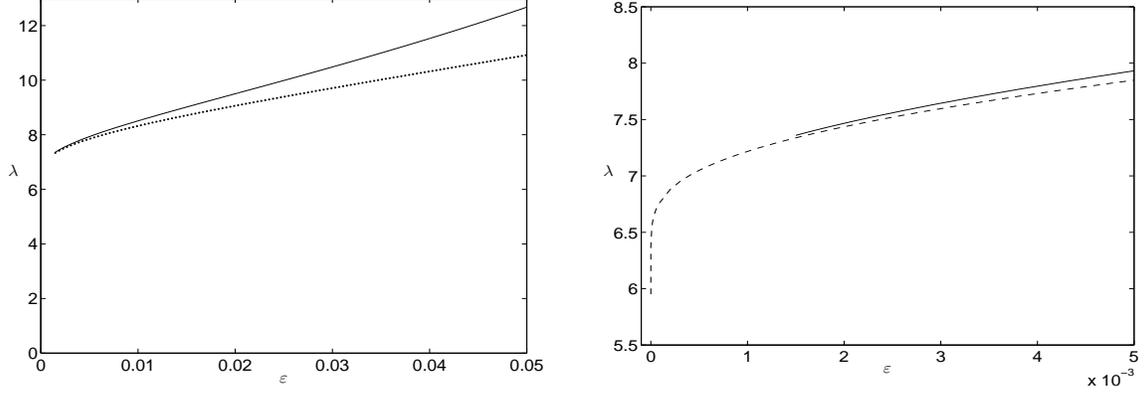


Figure 5. A quantitative test of Principal Result 4.2. The solid curve is the smallest eigenvalue calculated from the exact solution by a numerical evaluation of (4.39). The dotted curve is the asymptotic prediction determined from numerical solution of the transcendental equation (4.37). The right panel shows an enlargement for small values of  $\varepsilon$  and captures the limiting behavior  $\lambda^* \rightarrow \lambda_0$  as  $\varepsilon \rightarrow 0$ .

### 4.3 Weaker Bi-Laplace: $\delta \ll \mathcal{O}(\varepsilon^2)$

In this subsection we consider the case where  $\delta \ll \mathcal{O}(\varepsilon^2)$ . In terms of the inner variables  $y = \varepsilon^{-1}(x - x_0)$  and  $v(y) = u(x_0 + \varepsilon y)$ , the canonical inner solution satisfies

$$-\frac{\delta}{\varepsilon^2} \Delta_y^2 v_c + \Delta_y v_c + \varepsilon^2 \lambda v_c = 0, \quad y \in \mathbb{R}^2 \setminus \Omega_0; \quad v = \partial_n v = 0, \quad y \in \partial\Omega_0, \quad (4.41 a)$$

$$v_c \sim \log |y| + \chi, \quad \text{as } |y| \rightarrow \infty, \quad (4.41 b)$$

for some constant  $\chi$  to be determined. In our analysis below, where we allow  $\Omega_0$  to have an arbitrary shape, there are two regions that must be analyzed: The region where  $|y| = \mathcal{O}(1)$  is called the inner region, while the range of  $y$  where  $\text{dist}(y, \partial\Omega_0) = \mathcal{O}(\sqrt{\delta/\varepsilon^2}) \ll 1$  is called the sub-inner region.

As similar to the boundary layer analysis of §4.1, the leading order inner solution  $v_{c0}$  for (4.41), defined away from an  $\mathcal{O}(\sqrt{\delta/\varepsilon^2})$  neighborhood of  $\partial\Omega_0$ , must satisfy the effective boundary condition  $v_{c0} \rightarrow 0$  as  $y \rightarrow \partial\Omega_0$ . In the inner region, we expand  $v_c$  and  $\chi$  in (4.41) as

$$v_c = v_{c0} + \sqrt{\frac{\delta}{\varepsilon^2}} v_{c1} + \dots, \quad \chi \sim \chi_0 + \sqrt{\frac{\delta}{\varepsilon^2}} \chi_1 + \dots \quad (4.42)$$

Upon substituting (4.42) into (4.41), we obtain the leading-order problem

$$\Delta v_{c0} = 0, \quad y \in \mathbb{R}^2 \setminus \Omega_0; \quad v_{c0} = 0, \quad y \in \partial\Omega_0, \quad (4.43 a)$$

$$v_{c0} \sim \log |y| - \log d_0, \quad \text{as } |y| \rightarrow \infty; \quad \chi_0 \equiv -\log d_0. \quad (4.43 b)$$

In addition, assuming that  $\delta \gg \mathcal{O}(\varepsilon^6)$ , the problem for  $v_{c1}$  is

$$\Delta v_{c1} = 0, \quad y \in \mathbb{R}^2 \setminus \Omega_0; \quad v_{c1} \sim \chi_1, \quad \text{as } |y| \rightarrow \infty. \quad (4.44)$$

In our analysis below of the sub-inner layer near  $\partial\Omega_0$  we will derive an effective boundary condition for  $v_{c1}$  on  $\partial\Omega_0$ . Then, by imposing that  $v_{c1}$  is bounded as  $|y| \rightarrow \infty$  we will identify the constant  $\chi_1$  in (4.44).

The PDE (4.43) is a classical problem in electrostatics (cf. [23], [20]), where the constant  $d_0$  is referred to as the logarithmic capacitance of  $\Omega_0$ . The logarithmic capacitance is known analytically for various shapes of  $\Omega_0$  (cf. [23], [20]). For the circular domain  $\Omega_0 = \{y \mid |y| \leq 1\}$  we have  $v_{c0} = \log |y|$  and  $d_0 = 1$ .

Next, we insert a sub-inner layer near  $\partial\Omega_0$  to satisfy the boundary condition  $\partial_n v_c = 0$  on  $\partial\Omega_0$ . As in §4.1, we use

the orthogonal boundary-fitted coordinates  $(\eta, s)$  so that (4.41) is transformed on the region  $\eta \leq 0$  to (see (4.3))

$$-\frac{\delta}{\varepsilon^2} \left[ \partial_{\eta\eta} - \frac{\kappa_0}{1 - \kappa_0\eta} \partial_\eta + \frac{1}{1 - \kappa_0\eta} \partial_s \left( \frac{1}{1 - \kappa_0\eta} \partial_s \right) \right]^2 v_c + \left[ \partial_{\eta\eta} - \frac{\kappa_0}{1 - \kappa_0\eta} \partial_\eta + \frac{1}{1 - \kappa_0\eta} \partial_s \left( \frac{1}{1 - \kappa_0\eta} \partial_s \right) \right] v_c + \varepsilon^2 \lambda v_c = 0, \quad (4.45)$$

subject to  $v_c = \partial_\eta v_c = 0$  on  $\eta = 0$ . Here  $\kappa_0 = \kappa_0(s)$  is the curvature of  $\partial\Omega_0$ , with  $\kappa_0 = 1$  when  $\Omega_0$  is the unit disk.

We introduce the sub-inner layer variables  $\hat{\eta}$  and  $w_c$  by

$$\hat{\eta} = \eta/\sigma, \quad v_c = \sigma w_c, \quad \text{where } \sigma = \sqrt{\frac{\delta}{\varepsilon^2}} \ll 1. \quad (4.46)$$

Then, we expand  $w_c = w_{c0} + \mathcal{O}(1)$  and obtain to leading order from (4.45) that

$$-\partial_{\hat{\eta}\hat{\eta}\hat{\eta}\hat{\eta}} w_{c0} + \partial_{\hat{\eta}\hat{\eta}} w_{c0} = 0, \quad \hat{\eta} \leq 0; \quad w_{c0} = \partial_{\hat{\eta}} w_{c0} = 0, \quad \hat{\eta} = 0. \quad (4.47 a)$$

The leading-order matching condition to the inner solution is that

$$w_{c0} \sim -\hat{\eta} \partial_n v_{c0} \Big|_{\partial\Omega_0}, \quad \text{as } \hat{\eta} \rightarrow -\infty, \quad (4.47 b)$$

where  $\partial_n$  is the outward normal derivative to  $\partial\Omega_0$ . The solution to (4.47) is  $w_{c0} = -\partial_n v_{c0} \Big|_{\partial\Omega_0} [1 + \hat{\eta} - e^{\hat{\eta}}]$ . In this way, the leading-order solution in the sub-inner region is

$$v_c \sim \sqrt{\frac{\delta}{\varepsilon^2}} (w_{c0} + \dots) = -\sqrt{\frac{\delta}{\varepsilon^2}} \left[ \partial_n v_{c0} \Big|_{\partial\Omega_0} (1 + \hat{\eta} - e^{\hat{\eta}}) + \mathcal{O}(1) \right]. \quad (4.48)$$

Next, we employ a higher order matching condition between the far-field behavior of (4.46) as  $\hat{\eta} \rightarrow -\infty$  and the near-field behavior as  $\eta \rightarrow 0$  of the inner expansion (4.42). This yields the effective boundary condition

$$v_{c1} = -\partial_n v_{c0} \Big|_{\partial\Omega_0}, \quad y \in \partial\Omega_0. \quad (4.49)$$

The problem for  $v_{c1}$  is (4.44), subject to the boundary condition (4.49) and with  $v_{c1}$  bounded as  $|y| \rightarrow \infty$ . To determine  $\chi_1 = \lim_{|y| \rightarrow \infty} v_{c1}$ , we apply Green's identity to  $v_{c0}$  and  $v_{c1}$  over the region  $\Omega_L \setminus \Omega_0$ , where  $\Omega_L = \{y \mid |y| \leq L\}$ , and then pass to the limit  $L \rightarrow \infty$ . This yields that  $\int_{\partial\Omega_0} (-v_{c0} \partial_n v_{c1} + v_{c1} \partial_n v_{c0}) ds = -\lim_{L \rightarrow \infty} \int_{\partial\Omega_L} (v_{c0} \partial_n v_{c1} - v_{c1} \partial_n v_{c0}) ds$ . Since  $v_{c0} = 0$  and  $v_{c1} = -\partial_n v_{c0}$  on  $\partial\Omega_0$ , while  $v_{c0} \sim \log L$ ,  $v_{c1} \sim \chi_1$  and  $\partial_n v_{c1} = \mathcal{O}(L^{-2})$  on  $\partial\Omega_L$ , we get that

$$\chi_1 = -\frac{1}{2\pi} \int_{\partial\Omega_0} (\partial_n v_{c0})^2 ds. \quad (4.50)$$

In this way, the two-term expansion for  $\chi$  in (4.41), valid for an arbitrarily-shaped hole  $\Omega_0$ , is

$$\chi \sim -\log d_0 - \frac{1}{2\pi} \sqrt{\frac{\delta}{\varepsilon^2}} \int_{\partial\Omega_0} (\partial_n v_{c0})^2 ds + \dots. \quad (4.51 a)$$

It is then convenient to define  $d_\delta$  by  $\chi \equiv -\log d_\delta$  so that

$$d_\delta = e^{-\chi} \sim d_0 \left( 1 + \frac{1}{2\pi} \sqrt{\frac{\delta}{\varepsilon^2}} \int_{\partial\Omega_0} (\partial_n v_{c0})^2 ds \right). \quad (4.51 b)$$

Here  $d_0$  is the logarithmic capacitance of  $\Omega_0$  defined in terms of  $v_{c0}$  by (4.43). For the unit disk  $\Omega_0 = \{y \mid |y| \leq 1\}$ , we have  $v_{c0} = \log |y|$ ,  $d_0 = 1$ , and hence  $\chi \sim -\sqrt{\delta/\varepsilon^2}$  from (4.51 a). If we set  $\delta = \delta_0 \varepsilon^2$  in this last expression, we recover  $\chi \sim -\sqrt{\delta_0}$  as was previously derived in (4.21 b) of §4.2 for the small  $\delta_0$  limit for the critical scaling regime  $\delta = \delta_0 \varepsilon^2$ .

Finally, with  $d_\delta$  determined as in (4.51 b), we can then simply repeat the analysis in (4.32)–(4.36) involving the matching of the inner and outer solutions, which yields an approximation to the eigenvalue of (4.1). In place of Principal Result 4.2, we obtain the following result:

**Principal Result 4.3:** For  $\mathcal{O}(\varepsilon^6) \ll \delta \ll \mathcal{O}(\varepsilon^2)$ , there is an eigenvalue  $\lambda$  of (4.1) with  $\lambda \sim \lambda^*(\nu) + t.s.t.$ , where  $\lambda^*(\nu)$

satisfies the transcendental equation

$$R_h(x_0; \lambda^*) = -\frac{1}{2\pi\nu_\delta}, \quad \nu_\delta \equiv -1/\log(\varepsilon d_\delta), \quad d_\delta \sim d_0 \left( 1 + \frac{1}{2\pi} \sqrt{\frac{\delta}{\varepsilon^2}} \int_{\partial\Omega_0} (\partial_n v_{c0})^2 ds \right). \quad (4.52)$$

Here  $d_0$  is the logarithmic capacitance of  $\Omega_0$  defined in terms of  $v_{c0}$  by (4.43), and  $R_h$  is the regular part of the Helmholtz Green's function defined in (4.36). For the unit disk  $\Omega_0 = \{y \mid |y| \leq 1\}$ , we have  $d \sim 1 + \sqrt{\delta/\varepsilon^2}$ .

We remark that the leading-order term  $d \sim d_0$  still holds even without the extra technical restriction that  $\delta \gg \mathcal{O}(\varepsilon^6)$ . Using the leading-order result  $d \sim d_0$ , Principal Result 4.3 shows that when  $\delta \ll \mathcal{O}(\varepsilon^2)$  the approximation to the eigenvalue for (4.1) is the same to within all logarithmic terms as that for an eigenvalue of the Laplacian in a domain with a hole, as analyzed in [23]. The second-order term for  $d_\delta$  in (4.52) characterizes the weak influence on  $\lambda^*$  of the fourth-order term  $-\delta\Delta^2$  in the regime  $\mathcal{O}(\varepsilon^6) \ll \delta \ll \mathcal{O}(\varepsilon^2)$  for an arbitrarily-shaped hole.

#### 4.4 Stronger Bi-Laplace: $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$

Next, we consider the range  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$ . In terms of the inner variable  $y = \varepsilon^{-1}(x - x_0)$  and  $v(y) = u(x_0 + \varepsilon y)$ , the canonical inner solution still satisfies (4.41). In our analysis there are two regions that must be analyzed: an inner region with  $|y| = \mathcal{O}(1)$  where  $\Delta^2$  dominates in (4.41), and a far-inner region with  $z = y/(\sqrt{\delta}/\varepsilon) = \mathcal{O}(1)$  where  $-\Delta^2$  and  $\Delta$  balance in (4.41). We will first consider the special case of a circular hole where  $\Omega_0 = \{y \mid |y| \leq 1\}$ .

In the far-inner region we introduce the local variables  $z$  and  $w_c$ , and we obtain to leading order from (4.41) that

$$-\Delta_z^2 w_{c0} + \Delta w_{c0} = 0, \quad |z| \geq 0. \quad (4.53)$$

We look for a radially symmetric solution to (4.53) satisfying  $w_{c0} \sim \log|z|$  as  $|z| \rightarrow \infty$  and  $w_{c0} \rightarrow 0$  as  $|z| \rightarrow 0$ . The necessity of this second condition, which allows for an asymptotic matching of  $w_{c0}$  as  $|z| \rightarrow 0$  with the far-field behavior of the leading-order inner solution, is discussed in Remark 1. Any radially symmetric solution to (4.53) is a linear combination of  $\{1, \log|z|, K_0(|z|), I_0(|z|)\}$ . Upon using the asymptotics (4.20) for  $K_0(|z|)$  as  $|z| \rightarrow 0$ , it readily follows that the solution to (4.53) that satisfies  $w_{c0} \sim \log|z|$  as  $z \rightarrow \infty$  and  $w_{c0} \rightarrow 0$  as  $|z| \rightarrow 0$  is

$$w_{c0} = \gamma_e - \log 2 + \log|z| + K_0(|z|). \quad (4.54)$$

Upon using (4.20), we obtain the more refined behavior

$$w_{c0} \sim -\frac{1}{4}|z|^2 \log|z| + \frac{|z|^2}{4}(\log 2 + 1 - \gamma_e), \quad \text{as } |z| \rightarrow 0. \quad (4.55)$$

In terms of the inner variable  $y$ , defined by  $z = y/(\sqrt{\delta}/\varepsilon)$ , (4.55) then yields

$$w_{c0} \sim -\frac{|y|^2 \varepsilon^2}{4\delta} \log\left(\frac{\varepsilon}{\sqrt{\delta}}\right) + \frac{\varepsilon^2}{4\delta} [-|y|^2 \log|y| + |y|^2(\log 2 + 1 - \gamma_e)] + \dots. \quad (4.56)$$

The matching condition is that the far-field behavior as  $|y| \rightarrow \infty$  of the inner solution must agree with (4.56).

Motivated by (4.56), which suggests the asymptotic gauge functions, the inner solution for (4.41) is expanded as

$$v_c \sim \left( -\frac{\varepsilon^2}{\delta} \log\left(\frac{\varepsilon}{\sqrt{\delta}}\right) \right) v_{c0} + \frac{\varepsilon^2}{\delta} v_{c1} + \dots. \quad (4.57)$$

Upon substituting (4.57) into (4.41), and by using the matching condition (4.56), we obtain that the radially symmetric functions  $v_{c0}(|y|)$  and  $v_{c1}(|y|)$  satisfy

$$\Delta_y^2 v_{c0} = 0, \quad \text{in } |y| \geq 1; \quad v_{c0} = v'_{c0} = 0, \quad \text{on } |y| = 1; \quad v_{c0} \sim |y|^2/4, \quad \text{as } |y| \rightarrow \infty, \quad (4.58 a)$$

and

$$\Delta_y^2 v_{c1} = 0, \quad \text{in } |y| \geq 1; \quad v_{c1} = v'_{c1} = 0, \quad \text{on } |y| = 1, \quad (4.58 \text{ b})$$

$$v_{c1} \sim -\frac{|y|^2}{4} \log |y| + \frac{|y|^2}{4} (\log 2 + 1 - \gamma_e), \quad \text{as } |y| \rightarrow \infty. \quad (4.58 \text{ c})$$

These two problems can be solved explicitly to obtain

$$v_{c0} = \frac{1}{4} (|y|^2 - 1) - \frac{1}{2} \log |y|; \quad v_{c1} = \frac{1}{4} (\log 2 + 1 - \gamma_e) (|y|^2 - 1) - \frac{|y|^2}{4} \log |y| - \left( \frac{1}{2} \log 2 + \frac{1}{4} - \frac{\gamma_e}{2} \right) \log |y|. \quad (4.59)$$

**Remark 1** *The condition  $w_{c0} \rightarrow 0$  as  $|z| \rightarrow 0$  for the solution (4.53) is required in order to match to the inner solution. If instead,  $w_{c0} \rightarrow A \neq 0$  as  $|z| \rightarrow 0$ , then in the inner region we would expand  $v_c \sim v_{c0}(|y|) + \mathcal{O}(1)$ , to obtain the leading-order problem  $\Delta_y^2 v_{c0} = 0$  in  $|y| \geq 1$  with  $v_{c0} = v'_{c0} = 0$  on  $|y| = 1$  with  $v_{c0} \rightarrow A$  as  $|y| \rightarrow \infty$ . Since  $v_{c0}$  is a linear combination of  $\{1, \log |y|, |y|^2, |y|^2 \log |y|\}$ , there is no such solution when  $A \neq 0$ .*

Next, to determine the matching behavior as  $x \rightarrow x_0$  required for the outer solution, we let  $|z| \rightarrow \infty$  in (4.54), and write the resulting expression in terms of  $y$  using  $|z| = y / (\sqrt{\delta}/\varepsilon)$ . This yields that

$$v_c \sim \log |y| - \log d_\infty + \mathcal{O}(1), \quad \text{as } |y| \rightarrow \infty; \quad d_\infty \equiv \frac{2\sqrt{\delta}}{\varepsilon} e^{-\gamma_e}. \quad (4.60)$$

With the far-field behavior of the solution to (4.41) now known for the regime  $\mathcal{O}(\varepsilon^2) \ll \delta \ll 1$ , we then simply repeat the analysis in (4.32)–(4.36) for the matching to the outer solution. This leads to the following result:

**Principal Result 4.4:** *Consider (4.1) with a circular hole of radius  $\varepsilon$  centered at  $x = x_0$  for the range  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$ . Then, there is an eigenvalue  $\lambda$  of (4.1) with  $\lambda \sim \lambda^*(\nu_\infty)$ , where  $\lambda^*(\nu_\infty)$  satisfies the transcendental equation*

$$R_h(x_0; \lambda^*) = -\frac{1}{2\pi\nu_\infty}, \quad \nu_\infty \equiv -1 / \log \left( 2\sqrt{\delta} e^{-\gamma_e} \right). \quad (4.61)$$

Here  $R_h$  is the regular part of the Helmholtz Green's function defined in (4.36) and  $\gamma_e$  is Euler's constant.

The central implication of this result is that on the range  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$ , the approximation  $\lambda^*$  to the eigenvalue is independent of the radius  $\varepsilon$  of the hole, and depends on  $\delta$ , the hole location  $x_0$ , and the shape of  $\Omega$ . Therefore, we state that it is in this asymptotic range of  $\delta$  where the asymptotic result for  $\lambda$  provides a smooth transition to the result obtained in §3 involving a point constraint. More precisely, the result (4.61) was previously obtained in (3.15) from formally taking the small  $\delta$  limit of the point constraint formulation of §3. Alternatively, if we formally set  $\delta = \varepsilon^2 \delta_0$  in (4.61) we recover the same limiting expression as that obtained by letting  $\delta_0 \gg 1$  in the asymptotic result (4.37) of Principal Result 4.2 derived for the critical scaling regime  $\delta = \mathcal{O}(\varepsilon^2)$ .

Although (4.61) does provide the leading asymptotic behavior as  $\delta \rightarrow 0$ , the example below for the annulus shows that it is not particularly accurate for moderately small  $\delta$ . The discrepancy at moderately small values of  $\delta$  arises from neglecting the boundary layer of width  $\mathcal{O}(\delta^{1/2})$  in the vicinity of  $\partial\Omega$  in which the clamped boundary conditions (1.3 b) are satisfied. For an arbitrary domain with smooth boundary, we now extend the analysis leading to Principal Result 4.4 to account for the boundary layer near  $\partial\Omega$ . Motivated by the analysis in §4.1, we substitute the expansion

$$u = u^*(x, \nu_\infty) + \delta^{1/2} u_1(x, \nu_\infty) + \cdots, \quad \lambda = \lambda^*(\nu_\infty) + \delta^{1/2} \lambda_1(\nu_\infty) + \cdots, \quad (4.62)$$

into (1.3) and equate powers of  $\delta^{1/2}$ . At leading order,  $(\lambda^*, u^*)$  satisfy (4.34), with normalization condition  $\int_\Omega [u^*]^2 dx = 1$ , while the pair  $(\lambda_1, u_1)$  satisfies

$$\Delta u_1 + \lambda^* u_1 = -\lambda_1 u^*, \quad x \in \Omega \setminus \{x_0\}; \quad u_1 = \partial_n u^*, \quad x \in \partial\Omega; \quad \int_\Omega u_1 u^* dx = 0. \quad (4.63)$$

The boundary condition for  $u_1$  on  $\partial\Omega$  in (4.63) arises from a similar boundary layer analysis as in §4.1 (see (4.12 b)).

To establish a condition that fixes  $\lambda_1$ , the singular behavior of  $u_1$  as  $x \rightarrow x_0$  is obtained by writing  $u_1 =$

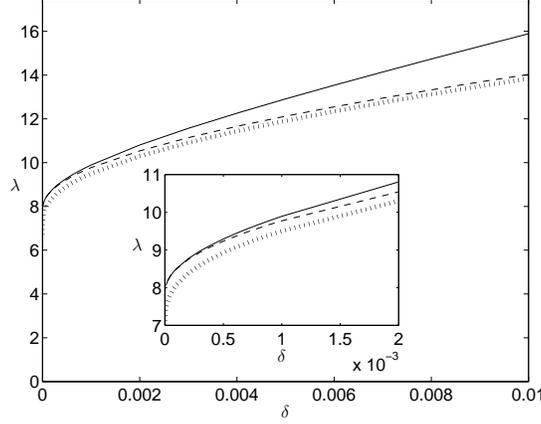


Figure 6. Comparison of the asymptotic result (4.68) for the lowest eigenvalue  $\lambda$  versus  $\delta$  (dotted curve) and the exact result (solid curve), as obtained by solving (4.39) numerically, for an annular domain  $\varepsilon \leq |x| \leq 1$  with  $\varepsilon = 0.01$ . The dashed curve is the asymptotic result (4.70) that represents adding a boundary layer correction term to the result for  $\lambda^*(\nu)$  of Principal Result 4.2 that applies for the regime  $\delta = \mathcal{O}(\varepsilon^2)$ . As expected, this latter result is more accurate than (4.68) when  $\delta$  is small, and it tends to the result in (4.68) as  $\delta$  is increased.

$A_1(\nu_\infty)\nu_\infty v_c(|y|)$  in the inner region. Using the established behavior of  $v_c$  in (4.60), we have that  $u_1$  must satisfy

$$u_1 \sim A_1 \nu_\infty \log |x - x_0| + A_1, \quad \text{as } x \rightarrow x_0. \quad (4.64)$$

We then decompose  $u_1 = u_{1p} + u_{1s}$  where  $u_{1p}$  is a smooth particular solution, and  $u_{1s}$  is a solution that is singular as  $x \rightarrow x_0$ . The local behavior (4.64) indicates that  $u_{1s} = -2\pi A_1 \nu_\infty G_h(x; x_0, \lambda^*)$ , leaving  $u_{1p}$  to solve

$$\Delta u_{1p} + \lambda^* u_{1p} = -\lambda_1 u^*, \quad x \in \Omega; \quad u_{1p} = \partial_n u^*, \quad x \in \partial\Omega. \quad (4.65)$$

The local behavior of the decomposed solution as  $x \rightarrow x_0$  is obtained by using (4.36 b) and the fact that  $u_{1p}$  is smooth for all  $x \in \Omega$ . This yields,

$$u_1 \sim u_{1p}(x_0) + A_1 \nu_\infty \log |x - x_0| - 2\pi A_1 \nu_\infty R_h(x_0, \lambda^*) + \mathcal{O}(1), \quad x \rightarrow x_0. \quad (4.66)$$

Upon comparing (4.66) and (4.64), and noting that  $-2\pi \nu_\infty R_h(x_0, \lambda^*) = 1$  from (4.61), we conclude that  $u_{1p}(x_0) = 0$ . We now show that this condition determines  $\lambda_1$ . By applying Green's identity to  $u_{1p}$  and  $G_h$  we derive

$$\int_{\Omega} [u_{1p}(\Delta G_h + \lambda^* G_h) - G_h(\Delta u_{1p} + \lambda^* u_{1p})] dx = \int_{\partial\Omega} (u_{1p} \partial_n G_h - G_h \partial_n u_{1p}) ds.$$

Upon substituting (4.63) and (4.36) into this identity, we obtain that  $-u_{1p}(x_0) + \lambda_1 \int_{\Omega} u^* G_h dx = \int_{\partial\Omega} (\partial_n u^*) (\partial_n G_h) ds$ . Finally, applying the condition  $u_{1p}(x_0) = 0$  and recalling from (4.35) that  $u^* = -2\pi \mathcal{C} G_h(x; x_0, \lambda^*)$ , we conclude that

$$\lambda_1 = \frac{\int_{\partial\Omega} (\partial_n G_h)^2 ds}{\int_{\Omega} (G_h)^2 dx}. \quad (4.67)$$

The constant  $A_1$  is  $A_1 = -[2\pi \nu_\infty]^{-1} \int_{\Omega} G_h u_{1p} dx / \int_{\Omega} G_h^2 dx$ , which results from imposing the normalization condition  $\int_{\Omega} u^* u_1 dx = 0$ . We summarize the result as follows:

**Principal Result 4.5:** Consider (4.1) with a circular hole of radius  $\varepsilon$  centered at  $x = x_0$  for the range  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$ . Then, there is an eigenvalue  $\lambda$  of (4.1) with

$$\lambda \sim \lambda^*(\nu_\infty) + \delta^{1/2} \lambda_1(\nu_\infty) + \dots, \quad (4.68)$$

where  $\lambda^*(\nu_\infty)$  satisfies the transcendental equation (4.61), and  $\lambda_1$  is given in (4.67).

As a test of the accuracy of Principal Result 4.5, we revisit the case considered at the end of §4.2 where  $\Omega$  is the

annular domain  $\varepsilon < r < 1$ , for which closed form solutions are available. The exact eigenvalues are determined by (4.39), which are solved numerically. To implement the asymptotic theory, we recall that  $R_h$  is given in (4.40). In addition, by calculating  $G_h$  analytically from (4.36), we determine  $\lambda_1$  in (4.67) as

$$\lambda_1(\nu_\infty) = \frac{4}{\pi^2} \left( \int_0^1 r \left[ Y_0(\sqrt{\lambda^*(\nu_\infty)r}) J_0(\sqrt{\lambda^*(\nu_\infty)}) - Y_0(\sqrt{\lambda^*(\nu_\infty)}) J_0(\sqrt{\lambda^*(\nu_\infty)r}) \right]^2 dr \right)^{-1}. \quad (4.69)$$

As  $\delta \rightarrow 0$ , for which  $\lambda^* \rightarrow \lambda_0$ , where  $\lambda_0$  is an eigenvalue of the Laplacian, (4.69) readily reduces upon using the Wronskian relation between  $J_0$  and  $Y_0$  to  $\lambda_1 \rightarrow 2\lambda_0$ . A similar boundary layer correction term can be added to the result for  $\lambda^*$  in (4.37) of Principal Result 4.2 that is valid when  $\delta = \mathcal{O}(\varepsilon^2)$ . This leads to the approximation

$$\lambda \sim \lambda^*(\nu) + \delta^{1/2} \lambda_1(\nu) + \dots. \quad (4.70)$$

Here  $\lambda^*(\nu)$  and  $\nu$  are defined in (4.37) and  $\lambda_1$  is obtained by replacing  $\nu_\infty$  in (4.69) with  $\nu$ .

For a circular hole with radius  $\varepsilon = 0.01$ , in Fig. 6 we show a reasonably favorable comparison over a range of values of  $\delta$  between the asymptotic results (4.68) and (4.70) for the lowest eigenvalue and the exact result, as obtained by solving (4.39) numerically. From the insert in this figure, we observe that (4.70) provides a more accurate prediction than (4.68) when  $\delta$  is small, and that the results in (4.70) and (4.68) are essentially indistinguishable for larger  $\delta$ . However, neither approximation is particularly accurate when  $\delta \approx 0.01$ . The reason for the discrepancy is likely due to not including higher order boundary layer contributions of order  $\mathcal{O}(\delta)$ . In fact, from (4.13), one of the terms of  $\mathcal{O}(\delta)$  is  $\lambda_0^2 \delta$  which is quantitatively significant even when  $\delta = 0.01$ .

We remark that, although we have derived Principal Results 4.4 and 4.5 only for the case of a circular hole, these results still hold for an arbitrarily-shaped hole  $\Omega_0$ . Therefore, the effect of the hole shape on the eigenvalue on the range  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$  is only through higher-order correction terms to (4.61) and (4.68). To show this, we observe that the leading-order solution for the far-inner region still holds, which in turn motivates the inner expansion (4.57). For an arbitrarily-shaped hole, and in place of (4.58), the non-radially symmetric functions  $v_{c0}$  and  $v_{c1}$  now satisfy

$$\Delta_y^2 v_{c0} = 0, \quad y \in \mathbb{R}^2 \setminus \Omega_0; \quad v_{c0} = \partial_n v_{c0} = 0, \quad y \in \partial\Omega_0, \quad (4.71 a)$$

$$v_{c0} \sim |y|^2/4, \quad \text{as } |y| \rightarrow \infty, \quad (4.71 b)$$

$$\Delta_y^2 v_{c1} = 0, \quad y \in \mathbb{R}^2 \setminus \Omega_0; \quad v_{c1} = \partial_n v_{c1} = 0, \quad y \in \partial\Omega_0, \quad (4.71 c)$$

$$v_{c1} \sim -\frac{|y|^2}{4} \log |y| + \frac{|y|^2}{4} (\log 2 + 1 - \gamma_e), \quad \text{as } |y| \rightarrow \infty. \quad (4.71 d)$$

Since the solutions of the homogeneous problem for  $v_{ck}$  for  $k = 0, 1$  are linear combinations of  $\{\rho^2 \log \rho, \rho^2, \log \rho, 1\}$ ,  $\{\rho^3, \rho \log \rho, \rho, \rho^{-1}\} \times \{\cos \theta, \sin \theta\}$ , and  $\{\rho^4, \rho^2, 1, \rho^{-2}\} \times \{\cos 2\theta, \sin 2\theta\}$  etc., where  $y = \rho(\cos \theta, \sin \theta)$  and  $\rho = |y|$ , the far-field behavior of the solution  $v_0$  to (4.71) must have the form

$$v_{c0} \sim \frac{1}{4}|y|^2 + A_0 \log |y| + f_0 \cdot y + \frac{y^T \mathcal{D}_0 y}{|y|^2} + o(1), \quad \text{as } |y| \rightarrow \infty, \quad (4.72)$$

for some constant  $A_0$ , vector  $f_0$ , and matrix  $\mathcal{D}_0$ , all determined by the shape of  $\Omega_0$ . Notice that we have imposed that  $|y|^{-1}(v_{c0} - |y|^2/4)$  is bounded as  $|y| \rightarrow \infty$ . In contrast, for  $v_{c1}$ , we must allow for a growth of order  $\mathcal{O}(y \log |y|)$  as  $|y| \rightarrow \infty$ . In terms of an arbitrary vector  $b_1$ , the far-field behavior of this solution to (4.71) has the form

$$v_{c1} \sim -\frac{1}{4}|y|^2 \log |y| + \frac{|y|^2}{4} (\log 2 + 1 - \gamma_e) + b_1 \cdot y \log |y| + f_1 \cdot y + A_1 \log |y| + \frac{y^T \mathcal{D}_1 y}{|y|^2} + o(1), \quad \text{as } |y| \rightarrow \infty, \quad (4.73)$$

for some constant  $A_1$ , vector  $f_1$ , and matrix  $\mathcal{D}_1$  determined in terms of the unknown  $b_1$  and the shape of  $\Omega_0$ . We remark that the unknown vector  $b_1$  is eventually determined by the gradient of the regular part of the Helmholtz Green's function at  $x = x_0$ . The terms  $A_j, f_j, \mathcal{D}_j$  for  $j = 0, 1$  then induce new higher-order non-radially symmetric correction terms to the far-inner solution. However, it is clear that the effect of these terms on the eigenvalue is of negligible asymptotic order as compared to (4.61).

## 5 A Numerical Experiment

In this section we illustrate Principal Results 4.4 and 4.5 for the case where  $\Omega$  is the unit disk with an off-centered hole of radius  $\varepsilon$  centered at  $x_0 \in \Omega$ . We will consider the parameter regime where  $\mathcal{O}(\varepsilon^2) \ll \delta \ll \mathcal{O}(1)$ .

We first must determine the regular part  $R_h(x_0; \lambda^*)$  of the Helmholtz Green's function defined in (4.36). Set  $x = r(\cos \theta, \sin \theta)$  and suppose without loss of generality that  $x_0 = (r_0, 0)$ . A standard Fourier series expansion of the solution to (4.36) yields

$$\begin{aligned} G_h(x_0; \lambda^*) &= -\frac{1}{4} J_0(r_{<} c) \left( Y_0(r_{>} c) - J_0(r_{>} c) \frac{Y_0(c)}{J_0(c)} \right) \\ &\quad - \frac{1}{2} \sum_{m=1}^{\infty} \cos(m\theta) J_m(r_{<} c) \left( Y_m(r_{>} c) - J_m(r_{>} c) \frac{Y_m(c)}{J_m(c)} \right), \end{aligned} \quad (5.1)$$

where we have defined  $r_{<}$ ,  $r_{>}$ , and  $c$  by

$$r_{<} \equiv \min(r, r_0), \quad r_{>} \equiv \max(r, r_0), \quad c \equiv \sqrt{\lambda^*}.$$

Similarly, the Fourier series expansion of the free-space Green's function is

$$\log|x - x_0| = \log r_{>} - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{r_{<}}{r_{>}} \right)^m \cos(m\theta). \quad (5.2)$$

By combining (5.1) and (5.2), and noting the singularity behavior in (4.36), we identify  $R_h$  as

$$\begin{aligned} R_h(x_0; \lambda^*) &= \frac{1}{2\pi} \log r_0 - \frac{1}{4} J_0(r_0 c) \left( Y_0(r_0 c) - J_0(r_0 c) \frac{Y_0(c)}{J_0(c)} \right) \\ &\quad + \sum_{m=1}^{\infty} \left\{ -\frac{1}{2\pi m} - \frac{1}{2} J_m(r_0 c) \left( Y_m(r_0 c) - J_m(r_0 c) \frac{Y_m(c)}{J_m(c)} \right) \right\}. \end{aligned} \quad (5.3)$$

The series in (5.3) is convergent as can be seen by using the large- $m$  asymptotics  $J_m(r) \sim \frac{(r/2)^m}{\Gamma(m+1)} (1 + O(1/m))$ ,  $Y_m(r) \sim (r/2)^{-m} \Gamma(m) (1 + O(1/m))$ , so that for large  $m$ , the terms of the sum in (5.3) behave like  $O(1/m^2)$ .

From Principal Result 4.4,  $\lambda^*$  is a root of the transcendental equation

$$R_h(x_0; \lambda^*) = \frac{1}{2\pi} \log \left( 2\sqrt{\delta} e^{-\gamma} \right). \quad (5.4)$$

We seek to determine the lowest root of (5.4). First, consider the case where  $r_0$  is small. Using the small- $r$  expansions  $J_m(r) \sim r^m 2^{-m} / \Gamma(1+m)$ ,  $m > 0$  and  $Y_m(r) \sim -r^m 2^m \Gamma(m) / \pi$ ,  $m > 0$ , one readily sees that the sum in (5.3) tends to zero as  $r_0 \rightarrow 0$ . Using the small- $r$  asymptotics of  $Y_0(r) \sim 2/\pi(\ln r - \ln 2 + \gamma)$  and  $J_0(r) \sim 1$  we readily obtain that

$$R_h(x_0, \lambda^*) \sim \frac{1}{2\pi} (\log 2 - \gamma - \log c) + \frac{1}{4} \frac{Y_0(c)}{J_0(c)}, \quad |x_0| \ll 1.$$

Therefore, for  $x_0 = 0$ , (5.4) reduces to

$$-\log c + \frac{\pi}{2} \frac{Y_0(c)}{J_0(c)} = \frac{1}{2} \log \delta. \quad (5.5)$$

More generally, for  $|x_0| \neq 0$ , expanding (5.4) to two orders yields the following two-term approximation  $\lambda^{*,2}$  to  $\lambda^*$ :

$$\sqrt{\lambda^{*,2}} \sim c_0 + \frac{\pi}{\log \left( 2e^{-\gamma} \sqrt{\delta} \right)} \frac{J_0^2(c_0 r_0) Y_0(c_0)}{2J_0'(c_0)}. \quad (5.6)$$

Here  $c_0 \approx 2.4048$  is the first root of  $J_0(c_0) = 0$ .

For  $\delta = 10^{-5}$  and hole radius  $\varepsilon = 0.001$ , in Fig. 7 we compare the one-term  $\lambda \sim c_0^2$ , two-term (5.6), and hybrid approximation (5.4) for the lowest eigenvalue of the (4.1) with the corresponding ‘‘exact’’ result as obtained by solving (4.1) numerically. The hybrid result is seen to provide a decent approximation to the eigenvalue.

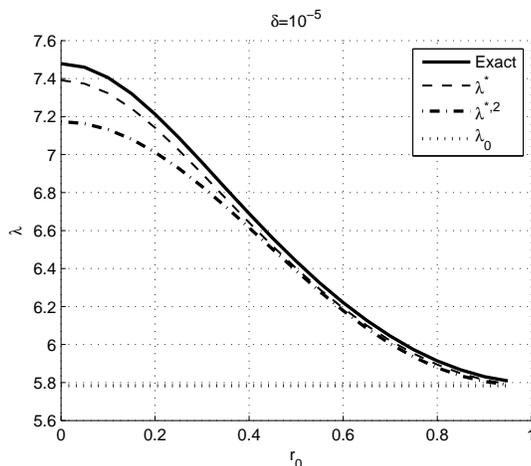


Figure 7. For  $\delta = 10^{-5}$  and hole radius  $\varepsilon = 0.001$ , we compare the one-term  $\lambda \sim c_0^2$ , two-term (5.6), and hybrid approximation (5.4) for the lowest eigenvalue of (4.1) with the corresponding “exact” result as obtained by solving (4.1) numerically. The domain is the unit disk with an off-centered hole at distance  $r_0$  from the origin.

Next, we show how to calculate the coefficient  $\lambda_1$ , given in (4.67) as needed in Principal Result 4.5. Upon using the Wronskian relation in (5.1), we calculate the numerator in (4.67) as

$$\int_{\partial\Omega} (\partial_n G)^2 ds = \frac{1}{2\pi} \sum_{m=0}^{\infty} \left[ \frac{J_m(cr_0)}{J_m(c)} \right]^2. \quad (5.7)$$

Next we compute the denominator in (4.67) as

$$\int_{\Omega} G^2 dx = -\frac{\pi}{2} \sum_{m=0}^{\infty} \int_0^1 (J_m(cr_{<}) Z_m(cr_{>}; m))^2 r dr, \quad \text{where} \quad Z_m(x; n) \equiv Y_m(x) - J_m(x) \frac{Y_n(c)}{J_n(c)}. \quad (5.8)$$

Then, by using the indefinite integral (cf. [13] formula 5.54)

$$\int B_m^2(cr) r dr = \frac{r^2}{2} (B_m^2(cr) - B_{m-1}(cr)B_{m+1}(cr)), \quad (5.9)$$

where  $B$  is any Bessel function, we obtain, after some simplifications, that

$$\begin{aligned} \int_0^1 (J_m(cr_{<}) Z_m(cr_{>}; m))^2 r dr &= J_m^2(cr_0) \left[ \frac{r_0^2}{2} Z_{m-1}(cr_0; m) Z_{m+1}(cr_0; m) + \frac{1}{2} Z_{m-1}(c; m) Z_{m+1}(c; m) \right] \\ &\quad - \frac{r_0^2}{2} Z_m^2(cr_0; m) J_{m-1}(cr_0) J_{m+1}(cr_0). \end{aligned}$$

This allows us to compute the denominator in (4.67).

In Table 1 we give some full numerical results for the lowest eigenvalue of (4.1). The computations were done assuming a hole radius of  $\varepsilon = \sqrt{\delta/100}$ . These “exact” results are then compared with the asymptotic result  $\lambda \sim \lambda^*$  and the improved asymptotic result  $\lambda \sim \lambda^* + \sqrt{\delta}\lambda_1$ , which adds the boundary layer correction term. From this table we observe that the improved asymptotic result agrees very favorably with the full numerical result. In each of the following examples, the “exact” results are obtained by means of finite element simulation [11] of the full problem (4.1).

In Fig. 8, we apply Principal Result 4.5 to the case of the disk but with the perturbing hole centered away from the origin. The inclusion of the  $\delta^{1/2}\lambda^*$  boundary correction term improves the accuracy of the expansion dramatically, even when  $\delta$  is moderately small. The full numerical simulations of (4.1) with an off centered hole are obtained from finite element simulation [11].

$r_0$	$\delta$	$\lambda_{exact}$	$\lambda^*$	$\lambda_1$	$\lambda^* + \sqrt{\delta}\lambda_1$	$(\lambda_{exact} - \lambda^*)/\sqrt{\delta}$
0	0.0001	8.106	7.912	16.418	8.076	19.430
0	1e-05	7.443	7.391	15.126	7.439	16.346
0	1e-06	7.088	7.073	14.369	7.087	15.118
0.2	0.0001	7.718	7.545	14.740	7.692	17.267
0.2	1e-05	7.188	7.141	14.022	7.185	14.650
0.2	1e-06	6.898	6.885	13.557	6.899	13.020
0.5	0.0001	6.693	6.563	11.625	6.679	12.936
0.5	1e-05	6.434	6.400	11.630	6.437	10.734

Table 1. For a single hole of radius  $\varepsilon = \sqrt{\delta/100}$ , we compare the asymptotic and numerical results for the lowest eigenvalue of (4.1) for various values of  $\delta$  and  $r_0$ , where  $r_0$  is the distance of the hole from the center of the unit disk. The third column is the full numerical result, the fourth column  $\lambda^*$  is the solution to (5.4), the fifth column is the boundary layer correction term  $\lambda_1$  in (4.67), while the sixth column is the asymptotic approximation  $\lambda \sim \lambda^* + \sqrt{\delta}\lambda_1$ , which includes the boundary layer correction term.

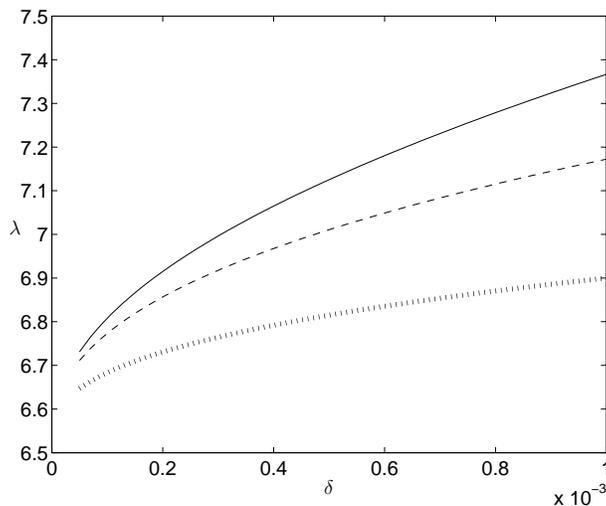


Figure 8. Application of Principal Result 4.5 for the case of a hole at  $x_0 = (0.5, 0)$  with radius  $\varepsilon = 0.01$ . The solid line is from full numerical simulation of (4.1) together with the leading order term (dotted line) and two term (dashed line) of the expansion (4.68).

## 6 Discussion

We have analyzed the limiting asymptotic behavior of the mixed eigenvalue problem (4.1) in the dual limit  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow 0$ . Our analysis has identified the following three key parameter regimes where different eigenvalue asymptotics occur:  $\delta = \mathcal{O}(\varepsilon^2)$ ,  $\delta \ll \mathcal{O}(\varepsilon^2)$ , and  $\mathcal{O}(\varepsilon^2) \ll \delta \ll 1$ . In the regime  $\mathcal{O}(\varepsilon^2) \ll \delta \ll 1$  we have shown in Principal Result 4.4 that the leading-order asymptotic behavior of an eigenvalue of (4.1) is asymptotically independent of  $\varepsilon$ . As a result, this regime provides a transition to the point constraint problem associated with the  $\delta = \mathcal{O}(1)$  regime. Results from the asymptotic theory were favorably compared with full numerical results.

There are several directions that warrant further investigation. Firstly, by developing a boundary integral method to compute  $\chi(\delta_0)$  numerically for an arbitrarily-shaped hole from the canonical inner problem (4.17) that holds for the scaling regime  $\delta = \mathcal{O}(\varepsilon^2)$ , Principal Result 4.2 could then be readily implemented for general hole shapes. In this paper,  $\chi(\delta_0)$  has been determined analytically only for a circular-shaped hole.

A second more fundamental open direction would be to analyze (4.1) for the regime  $\delta = \mathcal{O}(1)$  in the presence of  $N \geq 1$  holes of asymptotically small radii  $\mathcal{O}(\varepsilon) \ll 1$ . In the limit  $\varepsilon \rightarrow 0$ , and for  $\delta = \mathcal{O}(1)$ , an eigenvalue  $\lambda_\varepsilon$  of the

perturbed problem tends to an eigenvalue  $\lambda_0$  of the limiting point constraint problem

$$-\delta\Delta^2 u_0 + \Delta u_0 + \lambda_0 u = 0, \quad x \in \Omega; \quad \int_{\Omega \setminus \Omega_\varepsilon} u_0^2 dx = 1, \quad (6.1 a)$$

$$u_0 = \partial_n u_0 = 0, \quad x \in \partial\Omega; \quad u_0(x_j) = 0, \quad j = 1, \dots, N, \quad (6.1 b)$$

where  $x_j$  for  $j = 1, \dots, N$  are the centers of the small holes. It would be interesting to develop a numerical method to compute the eigenvalues  $\lambda_0$  of this point constraint problem, and to determine how they depend on  $\delta$ , with  $\delta = \mathcal{O}(1)$ , and the hole locations. In particular, where should the centers of  $N$  holes be located so as to minimize the principal eigenvalue of the limiting point constraint problem (6.1)? In the unit disk, optimal configurations of small holes that minimize the principal eigenvalue of the Laplacian in a 2-D domain  $\Omega$  with Neumann boundary condition on  $\partial\Omega$  and with a homogeneous Dirichlet boundary condition on each of the  $N$  holes were identified in [14]. For  $\delta = \mathcal{O}(1)$ , and for the case of multiple small holes, it would also be interesting to extend the analysis of [5] and [15], valid for a one hole pattern for the pure biharmonic operator, to determine  $\varepsilon$ -dependent correction terms for the difference  $\lambda_\varepsilon - \lambda_0$ .

Finally, it would be interesting to analyze localization behavior for the eigenfunctions  $u_0$  of the point constraint problem (6.1). For the pure biharmonic eigenvalue defined in a thin rectangular domain, and with a clamped point, the numerical computations in [10] showed that almost all of the eigenfunctions of the point constraint problem are typically confined to one side of a vertical line parallel to the thin edge of the rectangle that goes through the clamped point. In this way, the existence of a clamped point in the domain has a large effect on the geometric patterns and localization behavior of the eigenfunctions. It would be interesting to extend this analysis of [10] to (4.1).

### Acknowledgements

TK and MJW gratefully acknowledge grant support from NSERC (Canada).

### References

- [1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, 9th edition, New York, NY, Dover Publications.
- [2] C. Alves, P. Antunes, *The Method of Fundamental Solutions Applied to the Calculation of Eigensolutions for 2D Plates*, Int. Journ. Numer. Methods. Eng., **77**, (2009), pp. 177–194.
- [3] I. V. Andrianov, V. V. Danishevs'ky, A. L. Kalamkarov, *Asymptotic Analysis of Perforated Plates and Membranes: Part 1: Static Problems for Small Holes*, Int. J. Solids and Structures., **49**, (2012), pp. 298–310.
- [4] K. A. Burgemeister, C. H. Hansen, *Calculating Resonant Frequencies of Perforated Plates*, J. Sound Vibrat. **196**(4), (1996), pp. 387–399.
- [5] A. Campbell, S. A. Nazarov, *Asymptotics of Eigenvalues of a Plate with a Small Clamped Zone*, Positivity, **5**, (2001), pp. 275–295.
- [6] G. Chardon, L. Daudet, *Low-Complexity Computation of Plate Eigenmodes with Vekua Approximations and the Method of Particular Solutions*, Comput. Mech., **52**, (2013), pp. 983–992.
- [7] S. Choi, H. Jeong, T. Kim, K. Kim, K. Park, *Free Vibration Analysis of Perforated Plates Using Equivalent Elastic Properties*, J. Korean Nuclear Society, **30**(5), (1998), pp. 416–423.
- [8] C. V. Coffman, R. J. Duffin, *On the Fundamental Eigenfunctions of a Clamped Punctured Disk*, Adv. in Appl. Math. **13**, No. 2, (1992), pp. 142–151.
- [9] C. V. Coffman, R. J. Duffin, D. H. Shaffer, *The Fundamental Mode of Vibration of a Clamped Annular Plate is not of One Sign*, Constructive Approaches to Mathematical Models (Proc. Conf. in honor of R. J. Duffin, Pittsburgh, Pa., 1978), pp. 267–277, Academic Press, New York-London-Toronto, Ont. 1979.
- [10] M. Filoche, S. Mayboroda, *Strong Localization Induced by One Clamped Point in Thin Plate Vibrations*, Phys. Rev. Lett., **103**, 254301, (2009).
- [11] FlexPDE6, *PDE Solutions Inc.*, URL <http://www.pdesolutions.com>
- [12] M. Flucher, *Approximation of Dirichlet Eigenvalues on Domains with Small Holes*, J. Math. Anal. Appl., **193**(1), (1995), pp. 169–199.
- [13] Gradshteyn and Ryzhik, *Tables on integrals, series, and products*. Academic Press, 1965.
- [14] T. Kolokolnikov, M. Titcombe, M. J. Ward, *Optimizing the Fundamental Neumann Eigenvalue for the Laplacian in a Domain with Small Traps*, Europ. J. Appl. Math., **16**(2), (2005), pp. 161–200.

- [15] M.C. Kropinski, A.E. Lindsay, M.J. Ward, *Asymptotic Analysis of Localized Solutions to Some Linear and Nonlinear Biharmonic Eigenvalue Problems*, Studies in Appl. Math., **126**(4), (2011), pp. 347–408.
- [16] W. M. Lee, J. T. Chen, *Free Vibration Analysis of Circular Plates with Multiple Circular Holes using Indirect BIEM and Addition Theorem*, J. Appl. Mechanics, **78**(1), 011015, (2010).
- [17] A. E. Lindsay and M. J. Ward, *Asymptotics of some nonlinear eigenvalue problems modelling a MEMS Capacitor. Part II: Multiple Solutions and Singular Asymptotics*, Europ. J. Appl. Math., **22**(2), (2010), pp. 83–123.
- [18] P. Laurençot, C. Walker, *A fourth-order model for MEMS with clamped boundary conditions*. <http://arxiv.org/abs/1304.2296>.
- [19] S. Ozawa, *Singular Variation of Domains and Eigenvalues of the Laplacian*, Duke Math. J., **48**(4), (1981), pp. 767–778.
- [20] T. Ransford, *Potential Theory in the Complex Plane*, London Math. Soc. Stud. Texts 28, Cambridge Univ. Press, (1995), Cambridge, U.K.
- [21] J. N. Reddy, *Theory and Analysis of Plates and Shells*, (2007), CRC Press, Taylor and Francis.
- [22] G. Sweers, *When is the First Eigenfunction for the Clamped Plate Equation of Fixed Sign?*, Electronic J. Differ. Equ. Conf., **6**, Southwest Texas State Univ., San Marcos, Texas, (2001), pp. 285–296.
- [23] M. J. Ward, W. D. Henshaw, J. Keller, *Summing Logarithmic Expansions for Singularly Perturbed Eigenvalue Problems*, SIAM J. Appl. Math., **53**(3), (1993), pp. 799–828.
- [24] M. J. Ward, J. B. Keller, *Strong Localized Perturbations of Eigenvalue Problems*, SIAM J. Appl. Math., **53**(3), (1993), pp. 770–798.