Hot spots in crime model

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UCLA Model of hot spots in crime

• Originally proposed by Short, D’Orsogna, Pasour, Tita, Brantingham, Bertozzi, and Chayes, 2008 [The UCLA model]

• Crime is ubiquitous but not uniformly distributed
  - Some neighbourhoods are worse than others, leading to crime "hot spots"
  - Crime hotspots can persist for long time.

Fig. 1. Dynamic changes in residential burglary hotspots for two consecutive three-month periods beginning June 2001 in Long Beach, CA. These density maps were created using ArcGIS.

Figure taken from Short et al., *A statistical model of criminal behaviour*, 2008.
• Crime is temporally correlated:
  
  - Criminals often return to the spot of previous crime
  
  - If a home was broken into in the past, the likelihood of subsequent breaking increases
  
  - Example: graffiti "tagging"
Modelling criminal’s movement

- In the original model, biased Brownian motion was used to model criminal’s movement

- Our goal is to extend this model to incorporate more realistic motion

- Typical human motion consists short periods of fast movement [car trips] interspersed with long periods of slow motion [pacing, thinking about theorems, sleeping...]

- Such motion is often modelled using *Levi Flights*: At each time, the speed is chosen according to a *power-law distribution*; direction chosen at random:

\[ |y(t + \delta t) - y(t)| = \delta t X \]

where \( X \) is a power-law distribution whose distribution function is

\[ f(d) = C |d|^{-\mu} \]

- \( \mu \) is the power law exponent
  - In 1D, \( 1 < \mu \leq 3 \); in 2D, \( 1 < \mu \leq 4 \).
  - \( \mu = 3 \) corresponds to Brownian motion in one dimension.
González, Hidalgo, Barabási, *Understanding individual human mobility patterns*, *Nature* 2008, use cellphone data to suggest that human motion follows “truncated” Levi flight distribution with $\mu \approx 2.75$. 

- Brownian motion
- Levi flight motion
Discrete (cellular automata) model

- Two variables

\[ A_k(t) \equiv \text{attractiveness at node } k, \text{ time } t; \]
\[ N_k(t) \equiv \text{criminal density at node } k \]

- **Modelling attractiveness:** Attractiveness has static and dynamic component:

\[ A_k(t) \equiv A^0 + B_k(t). \]

\[ B_k(t + \delta t) = \left[ (1 - \hat{\eta}) B_k(t) + \frac{\hat{\eta}}{2} (B_{k-1} + B_{k+1}) \right] (1 - w\delta t) + \delta t A_k N_k \theta. \]

- "broken window effect"  
- decay rate  
- # of robberies

- \( 0 < \hat{\eta} < 1 \) is the strength of broken window effect

- \( w \) is the decay rate
• Modelling criminal movement: Define the **relative weight** of a criminal moving from node $i$ to node $k$, where $i \neq k$, as

$$w_{i \rightarrow k} = \frac{A_k}{l^\mu |i - k|^\mu}. \quad (1)$$

- $l$ is the grid spacing, $\mu$ the Levi flight power law exponent
- The weight is **biased** by attractiveness field

• The **transition probability** of a criminal moving from point $i$ to point $k$, where $i \neq k$, is

$$q_{i \rightarrow k} = \frac{w_{i \rightarrow k}}{\sum_{j \in \mathbb{Z}, j \neq i} w_{i \rightarrow j}}. \quad (2)$$

• Update rule for criminal density:

$$N_k(t + \delta t) = \sum_{i \in \mathbb{Z}, i \neq k} N_i \cdot (1 - A_i \delta t) \cdot q_{i \rightarrow k} + \Gamma \delta t. \quad (3)$$

- $A_i \delta t \equiv$ probability that criminal robs
- $(1 - A_i \delta t) \equiv$ probability that no robbery occurs
- $N_i \cdot (1 - A_i \delta t) \equiv$ expected number of criminals at node $i$ that don’t rob
- $N_i \cdot (1 - A_i \delta t) \cdot q_{i \rightarrow k} \equiv$ expected number of criminals that move from mode $i$ to mode $k$.
- $\Gamma \delta t \equiv$ constant ”feed rate” of the criminals
Take a limit \( l, \delta t \ll 1 \):

- Main trick is to write \( A_i \sim A(x) \) where \( x = li \); then

\[
\sum_{j \in \mathbb{Z}, j \neq i} w_{i \to j} = \sum_{j \in \mathbb{Z}, j \neq i} \frac{A_j}{l^\mu |i - j|^\mu} = \sum_{j \in \mathbb{Z}, j \neq i} \frac{A_j - A_i}{l^\mu |i - j|^\mu} + \sum_{j \in \mathbb{Z}, j \neq i} \frac{A_i}{l^\mu |i - j|^\mu}
\]

\[
\sim \frac{1}{l} \int_{-\infty}^{\infty} \frac{A(y) - A(x)}{|x - y|^\mu} dy + l^{-\mu}2\zeta(\mu)A(x)
\]

- We recognize the integral as fractional Laplacian,

\[
\Delta^s f(x) = 2^{2s} \frac{\Gamma(s + 1/2)}{\pi^{1/2} |\Gamma(-s)|} \int_{-\infty}^{\infty} \frac{f(x) - f(y)}{|x - y|^{2s+1}} dy, \quad 0 < s \leq 1.
\]

- Key properties:
  - The normalization constant is chosen so that the Fourier transform is:

\[
\mathcal{F}_{x \to q} \{\Delta^s f(x)\} = -|q|^{2s} \mathcal{F}_{x \to q} \{f(x)\}.
\]

- \( s = 1 \) corresponds to the usual Laplacian: \( \Delta^s f(x) = f_{xx} \) if \( s = 1 \).
Continuum model

The continuum limit of CA model becomes

\[
\frac{\partial A}{\partial t} = \eta A_{xx} - A + \alpha + A\rho. 
\] (6)

\[
\frac{\partial \rho}{\partial t} = D \left[ A \Delta^s \left( \frac{\rho}{A} \right) - \frac{\rho}{A} \Delta^s (A) \right] - A\rho + \beta 
\] (7)

where

\[
s = \frac{\mu - 1}{2} \in (0, 1]; \quad \eta = \frac{l^2 \hat{\eta}}{2\delta tw} ; \quad D = \frac{l^2 s \pi^{1/2} 2^{-2s} |\Gamma(-s)|}{\delta t z \Gamma (2s + 1) w} ; \quad \alpha = A_0/w; \quad \beta = \Gamma \theta/w^2. 
\]

- Separation of scales: if \( l, \delta t \ll 1 \) then

\[
D \eta^{-s} \gg 1; \quad 0 < s \leq 1. 
\] (8)

- The special case \( s = 1 (\mu = 3) \) corresponds to regular diffusion \( \Delta^1 f(x) = f_{xx} \).

  - We recover the UCLA model because:

    \[
    A \left( \frac{\rho}{A} \right)_{xx} - \frac{\rho}{A} A_{xx} = \left( \rho_x - 2 \frac{\rho}{A} A_x \right)_x 
    \]

  - Note that \( D \to \infty \) as \( s \to 1^- \) since \(|\Gamma(-s)| \sim 1/(1 - s)\).
Simulation of continuum model

• Use a spectral method in space combined with method of lines in time.

• That is, we first discretize in space $x \in [0, L]$. To approximate $\Delta^s u$, we make use of Fourier transform:

$$\Delta^s u = \mathcal{F}^{-1} \left( -|q|^{2s} \mathcal{F}_{x \rightarrow q} \{u\} \right). \tag{9}$$

• This becomes FFT on a bounded interval

• **Matlab code** to estimate the discretization of $\Delta^s u(x), \ x \in [0, 1]$:

```matlab
n = numel(u);
qu = 2*pi*[0:n/2-1, -n/2:-1]';
LaplaceSu = ifft(-q.^((2*s).*fft(u)));
```

• This implicitly imposes periodic boundary conditions on the solution.
Comparison: discrete vs. continuum

Example: Take $\mu = 2.5$, $n = 60$, $l = 1/60$, $\hat{\eta} = 0.1$, $\delta t = 0.01$, $A_0 = 1$, $\Gamma = 3$.

Then the continuum model gives $s = 0.75$, $\eta = 0.001388$, $D = 0.1828$, $\alpha = 1$, $\beta = 3$.

Discrete model is represented by dots; continuum model by solid curves. Blue is $A$, red is $\rho$. Two hot-spots form.
Turing instability analysis

\[
\frac{\partial A}{\partial t} = \eta A_{xx} - A + \alpha + A\rho, \quad \frac{\partial \rho}{\partial t} = D \left[ A \Delta^s \left( \frac{\rho}{A} \right) - \frac{\rho}{A} \Delta^s (A) \right] - A\rho + \beta
\]

Steady state:

\[
\bar{A} = \alpha + \beta; \quad \bar{\rho} = \frac{\beta}{\alpha + \beta}.
\]

Linearization:

\[
A(x, t) = \bar{A} + \phi e^{\lambda t} e^{ikx}, \quad \rho(x, t) = \bar{\rho} + \psi e^{\lambda t} e^{ikx}.
\]

(10a)

(10b)

Using the Fourier transform property, we have:

\[
\Delta^s e^{ikx} = -|k|^2 s e^{ikx}
\]

so the eigenvalue problem becomes

\[
\begin{bmatrix}
-\eta|k|^2 - 1 + \bar{\rho} & \bar{A} \\
\frac{2\bar{\rho}}{A}D|k|^{2s} - \bar{\rho} & -D|k|^{2s} - \bar{A}
\end{bmatrix}
\begin{bmatrix}
\phi \\
\psi
\end{bmatrix} = \lambda
\begin{bmatrix}
\phi \\
\psi
\end{bmatrix}.
\]

(11)

The dispersion relation is then given by

\[
\lambda^2 - \tau\lambda + \delta = 0
\]
where
\[
\tau = -D|k|^{2s} - \eta|k|^2 - \bar{A} - 1 + \bar{\rho}; \quad \delta = D|k|^{2s} (\eta |k|^2 + 1 - 3\bar{\rho}) + \eta |k|^2 \bar{A} + \bar{A}.
\]

Note that \( \tau < 0 \) so the steady state is stable iff \( \delta > 0 \) for all \( k \). Equilibrium is stable if \( \bar{\rho} < 1/3 \). If \( \bar{\rho} > 1/3 \) then equilibrium is unstable iff

\[
\bar{A} < D\eta^s x^s \left(-1 + \frac{3\bar{\rho}}{x + 1}\right)
\]  \hspace{1cm} (12)

where \( x \) is the unique positive root of
\[
x^2 + x(2 + 3\bar{\rho}(1 - s)/s) + 1 - 3\bar{\rho} = 0.
\]
Comparison with numerics
The effect of changing $s$ on dispersion relationship

(a) A stable regime

(b) An unstable regime

(c) Fractional diffusion leads to stability

(d) Fractional diffusion leads to unstability

(e) Fractional diffusion leads to stability and then instability

(f) A regime in which $|k_2| < |k_1| = 1$
Dominant instability \[\text{[biggest } \lambda\text{]}\]

- Recall that in terms of original gridsize \(l\) and time step \(\delta t\), we have:

\[
s = \frac{\mu - 1}{2} \in (0, 1]; \quad \eta = \frac{l^2 \hat{\eta}}{2 \delta t w}; \quad D = \frac{l^{2s} \pi^{1/2} 2^{-2s} |\Gamma(-s)|}{\delta t z \Gamma(2s + 1) w}
\]

so that \(\eta^{-s} D = O\left((1 - s)^{-1} (\delta t)^{s-1}\right) \gg 1, \quad 0 < s \leq 1\)

- For a physically relevant regime, the continuum model satisfies the key relationship

\[
\eta^{-s} D \gg 1. \quad (13)
\]

Change the variables \(k = x^{1/2} \eta^{-1/2}\) and let \(M = D \eta^{-s} \gg 1\). Then we obtain

\[
\tau = -M x^s - x^2 + \bar{\rho} - 1 - \bar{A}; \quad \delta = M x^s (x + 1 - 3\bar{\rho}) + x \bar{A} + \bar{A}.
\]

The fastest growing mode corresponds to the maximum of the dispersion curve:

\[
\lambda^2 - \tau \lambda + \delta = 0 \quad \text{and} \quad \lambda = \tau_x / \delta_x.
\]
Asymptotically, this becomes

\[ k_{\text{fastest}}(s) \sim \left[ \frac{s\bar{\rho}(\bar{A} - 2 + 6\bar{\rho})}{D\eta} \right]^{\frac{1}{2(s+1)}}, \quad D\eta^{-s} \gg 1. \]  

(14)

Expected number of “bumps” \( \approx \) floor \( \left( \frac{L}{2\pi k_{\text{fastest}}} \right) \).  

(15)

\( k_{\text{fastest}} \) is at a maximum when \( s \) satisfies

\[ \log \left( \frac{\bar{\rho}(\bar{A} - 2 + 6\bar{\rho})}{D\eta} s \right) = s + 1 \]
Comparison with numerics

\[ l = 0.01, \delta t = 0.05, \hat{\eta} = 0.02, A_0 = 1, \Gamma = 3 \]

- The initial instability has sinusoidal shape
• Eventually, hot-spot forms.
  - Hot-spots are localized regions which are \textit{not} of the sinusoidal shape!
  
  - In general, the total number of stable hot-spots \textit{does not} correspond to fastest-growing Turing mode!
  
  - The hot-spot regime is separate from the Turing regime!
Figure 7. Numerically computed bifurcation diagram of $A(0)$ vs. $\gamma$. The parameter values are $\alpha = 1, \varepsilon = 0.05, x \in [0, 1]$, and $D = 2$. A localized hot-spot appears for large values of $A(0)$. The asymptotics $A(0) \sim \frac{2(\gamma - \alpha)}{\varepsilon \pi}$ (see (2.19)) are shown by a dotted line. The constant steady state $A \sim \gamma$ is indicated by a solid straight line. Turing patterns are born from the spatially uniform steady state as a result of a Turing bifurcation at $\gamma \sim 3\alpha/2 = 1.5$. The weakly nonlinear regime is indicated by a dashed parabola coming out of the bifurcation point. Inserts shows the change in the shape of the profile $A(x)$ along the bifurcation curve.
Construction of hotspot solution

Hotspot solution satisfies:

\[
0 = \eta A_{xx} - A + \alpha + A\rho; \quad 0 = D \left[ A\Delta^s \left( \frac{\rho}{A} \right) - \frac{\rho}{A} \Delta^s (A) \right] - A\rho + \beta
\]  \hspace{1cm} (16)

and is periodic on \([-1, 1]\).

- **Key transformation:** Let \( \rho = vA^2 \); then

\[
0 = \eta A_{xx} - A + \alpha + A^3 v; \quad 0 = D \left[ A\Delta^s (vA) - vA\Delta^s (A) \right] - A^3 v + \beta
\]  \hspace{1cm} (17)

- **Inner problem:** Change variables \( x = \eta^{1/2}y \); then

\[
0 = A_{yy} - A + \alpha + A^3 v; \quad 0 = D\eta^{-s} \left[ A\Delta^s (vA) - vA\Delta^s (A) \right] - A^3 v + \beta
\]

- As before, \( D\eta^{-s} \gg 1 \) so that in the inner region,

\[
A\Delta^s_y (vA) - vA\Delta^s_y (A) \sim 0 \implies v(y) \sim \text{const.} \sim v_0
\]

- Change variables \( A = v_0^{-1/2}w(y) \), then

\[
w_{yy} - w + w^3 = 0 \implies w = \sqrt{2}\text{sech} (y)
\]
To determine $v_0$, integrate (17) and use the identity $\int f \Delta^s g - g \Delta^s f = 0$; then

$$\int A^3 v_0 \sim \int \beta$$

- The final result is

$$A(x) \sim \begin{cases} \frac{A_{\text{max}} w(x/\sqrt{\eta})}{\alpha}, & x = O(\varepsilon) \\ \alpha, & x \gg O(\varepsilon). \end{cases}$$

$$A_{\text{max}} \sim \frac{2l \beta \pi^{-3/2}}{\sqrt{\eta}}$$

where $l$ is the half-width of the spot.
Stability of hot-spots (1D, $s = 1$)

- **Localized states:** Consider a periodic pattern consisting of localized hotspots of radius $l$. It is stable iff $l > l_c$ where

$$l_c := \frac{(\eta D)^{1/4} \pi^{1/2} \alpha^{1/2}}{\beta^{3/4}}.$$  

- **Turing instability in the limit $\varepsilon \to 0$:**
  - Preferred Turing characteristic length:

$$l_{\text{turing}} \sim 2\pi \left[ \frac{D\eta}{\bar{\rho}(-2 + 3A + 6\bar{\rho})} \right]^{1/4}, \quad D\eta^{-1} \gg 1$$

- Note that both $O(l_c) = O(l_{\text{turing}}) = O((D\eta)^{1/4})$!
Example: $\alpha = 1, \quad \gamma = 2, \quad D = 1, \quad \varepsilon = 0.03$.

Then $l_{turing} = 0.60; \quad l_c = 0.13 < l_{turing}$
Small and large eigenvalues

- Near-translational invariance leads to “small eigenvalues (perturbation from zero)” corresponding eigenfunction is $\phi \sim w'$.
- Large eigenvalues are responsible for “competition instability”.
- Small eigenvalues become unstable before the large eigenvalues.
- Example: Take $l = 1, \gamma = 2, \alpha = 1, K = 2, \varepsilon = 0.07$. Then $D_{c,\text{small}} = 20.67$, $D_{c,\text{large}} = 41.33$.
  - if $D = 15 \implies$ two spikes are stable
  - if $D = 30 \implies$ two spikes have very slow developing instability
  - if $D = 50 \implies$ two spikes have very fast developing instability
Stability: large eigenvalues

- **Step 1:** Reduces to the nonlocal eigenvalue problem (NLEP):

\[ \lambda \phi = \phi'' - \phi + 3w^2 \phi - \chi \left( \int w^2 \phi \right) w^3 \quad \text{where } w'' - w + w^3 = 0. \]  \hspace{1cm} (18)

with

\[ \chi \sim \frac{3}{\int_{-\infty}^{\infty} w^3 dy} \left( 1 + \varepsilon^2 D \left( 1 - \cos \frac{\pi k}{K} \right) \frac{\alpha^2 \pi^2}{4l^4 \beta^3} \right)^{-1} \]

- **Step 2:** **Key identity:** \( L_0 w^2 = 3w^2 \), where \( L_0 \phi := \phi'' - \phi + 3w^2 \phi \). Multiply (18) by \( w^2 \) and integrate to get

\[ \lambda = 3 - \chi \int w^5 = 3 - \chi \frac{3}{2} \int w^3 \]

Conclusion: (18) is stable iff \( \chi > \frac{2}{\int w^3} \iff D > D_{c, \text{large}}. \)

- This NLEP in 1D can be fully solved!!
Stability: small eigenvalues

- Compute asymmetric spikes
- They bifurcate from symmetric branch
- The bifurcation point is precisely when $D = D_{c, \text{small}}$
- This is “cheating”... but it gets the correct threshold!!
Stability of $K$ spikes

- Possible boundary conditions:

<table>
<thead>
<tr>
<th>Config type</th>
<th>Boundary conditions for $\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single interior spike on $[-l, l]$</td>
<td>$\phi'(0) = 0 = \phi'(l)$</td>
</tr>
<tr>
<td>even eigenvalue</td>
<td></td>
</tr>
<tr>
<td>Single interior spike on $[-l, l]$</td>
<td>$\phi(0) = 0 = \phi'(l)$</td>
</tr>
<tr>
<td>odd eigenvalue</td>
<td></td>
</tr>
<tr>
<td>Two half-spikes at $[0, l]$</td>
<td>$\phi'(0) = 0 = \phi(l)$</td>
</tr>
<tr>
<td>$K$ spikes on $[-l, (2K - 1)l]$,</td>
<td>$\phi(l) = z\phi(-l)$, $\phi'(l) = z\phi'(-l)$,</td>
</tr>
<tr>
<td>Periodic BC</td>
<td>$z = \exp\left(\frac{2\pi ik}{K}\right)$, $k = 0 \ldots K - 1$</td>
</tr>
<tr>
<td>$K$ spikes on $[-l, (2K - 1)l]$,</td>
<td>$\phi(l) = z\phi(-l)$, $\phi'(l) = z\phi'(-l)$,</td>
</tr>
<tr>
<td>Neumann BC</td>
<td>$z = \exp\left(\frac{\pi ik}{K}\right)$, $k = 0 \ldots K - 1$</td>
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(same BC for $\psi$)
Two dimensions

Given domain of size \( S \), let

\[
K_c := 0.07037\eta^{-3/8}D^{-1/3} \left( \ln \frac{1}{\sqrt[3]{\eta}} \right)^{1/3} \beta \alpha^{-2/3} S.
\] (19)

Then \( K \) spikes are stable if \( K < K_c \). Example: \( \alpha = 1, \gamma = 2, \varepsilon = 0.08, D = 1 \).

We get \( S = 16, K_c \approx 10.19 \). Starting with random initial conditions, the end state consists of \( K = 7.5 < K_c \) hot-spots [counting boundary spots with weight \( 1/2 \) and corner spots with weight \( 1/4 \)], in agreement with the theory.
**Discussion**

- Natural Separation of scales: \( \eta^{-s} D \gg 1 \)
  - comes from the modelling assumptions
  - Required for hot-spot construction
  - The steady states are localized hotspots in the form of a sech, not sinusoidal bumps!

- Open question:
  - extend stability of hot-spots to Levi flights
  - More general models of human motion?

- There is an optimal Levi flight exponent \( 1 < \mu < 3 \) which “maximizes” the number of hot-spots. Do criminals “optimize” their strategy with respect to \( \mu \)?

- References: