Saturation in SIR model

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\begin{itemize}
  \item SIR model:
    \begin{align*}
    \frac{dS}{dt} &= -\frac{\beta}{N}SI; \\
    \frac{dI}{dt} &= \frac{\beta}{N}SI - \gamma I; \\
    \frac{dR}{dt} &= \gamma I
    \end{align*}
  \end{itemize}

  \begin{itemize}
  \item Here is US data:
  \end{itemize}

  \begin{itemize}
  \item Does SIR model fit coronavirus data?
    \begin{itemize}
    \item Think of $\beta = \beta(t)$ and $\gamma = \gamma(t)$.
    \item Estimate $\beta(t)$ $\gamma(t)$ from equations and data: $\beta(t) \approx -\frac{NS'(t)}{SI}$; $\gamma(t) \sim \frac{R'}{I}$
    \end{itemize}
  \end{itemize}
- Here is an example of plot of $\beta(t)$, $\gamma(t)$ from data:

- Note that $\gamma$ doesn’t change much, but $\beta(t)$ decreases steadily throughout the infection.

- The decrease of $\beta(t)$ is consistent for multiple datasets.

- Alternatively, we can plot $\beta I$ as a function of $I$:

- Note how $\beta I$ saturates as $I$ increases. This suggests that some saturation may occur.
Agent-based SIR model

- Assume there is an “infection radius” $r$; infection can only happen when susceptible comes within the distance $r$ of infected individual

- An infection occurs with some probability $p$ if a susceptible person comes within a radius $r$ of the infected person

- Assume people are “well-mixed”: all are located at random within a unit box.

- Infected individuals are removed with some rate $\gamma$. 
Let’s estimate $\beta \approx -\frac{S'(t)}{SI}$; $\gamma \sim \frac{R'}{I}$ we get this curve:

- Gamma is roughly constant $\approx 0.01$; but $\beta$ is clearly non-constant.
- There is a clear saturation of $\beta I$, when plotted versus $I$.
- This is similar to observed data for COVID in US, Canada, World and others.
- The reason is that when enough people are infected, they “cover” the whole area so introducing even more infecteds doesn’t
Continuum limit of ABM

- Given $I$ infected individuals, the probability of infection is: $p \ast E$, where $E$ is the probability that you are inside one of $I$ circles.

- $E$ is the fraction of the area occupied by the union of $I$ disks.

- For small $I$ and small $r$, we expect $E \approx aI$, where $a = \frac{\pi r^2}{A}$ is the area fraction for a single disk of radius $r$ inside a domain of area $A$.

- But as $I$ is increased, some circles intersect, so eventually $E$ saturates to 1.

- **Key question: what is the expected area fraction of $I$ disks?**
$E$: expected area fraction of union of $I$ disks

- Add $I$ areas, then subtract $\binom{I}{2}$ pairwise intersections, then add back $\binom{I}{3}$ triple intersections....

- $E = IE_1 - \binom{I}{2} E_2 + \binom{I}{3} E_3 + \ldots - (-1)^I E_I$ where:

  - red: 0.28854
  - brown: 0.06671
  - blue: 0.00348
  - yellow: 0.00086
- \( E_1 = a = \frac{\pi r^2}{A} \) is the area fraction of a single disk

- \( E_2 \) is the expected area fraction of intersection of two random disks of radius \( a \)

- etc

- **CLAIM:** \( E_j = a^j \)

  - Proof: either by direct integration, or by scaling argument: \( E_j = C (a)^j \) for small \( a \). Also \( E_j = 1 \) when \( a = 1 \), so \( C = 1 \).

- So we have obtain: \( E = - \sum_{j=1}^{I} \binom{I}{j} (-a)^j = 1 - (1 - a)^I \).

- **Conclusion:** \( E = 1 - (1 - a)^I \sim (1 - \exp (-Ia)) \), where \( a = \frac{\pi r^2}{A} \) is the area fraction for a single disk of radius \( r \) inside a domain of area \( A \).

- SIR model becomes:

\[
\begin{align*}
\frac{dS}{dt} &= -pS \left(1 - \exp (-Ia)\right); \\
\frac{dI}{dt} &= pS \left(1 - \exp (-Ia)\right) - \gamma I; \\
\frac{dR}{dt} &= \gamma I
\end{align*}
\]

(2)

- Note that for small \( aI \), we recover mass-action law: \( pS \left(1 - \exp (Ia)\right) \sim paSI \)
Direct comparison: ABM vs ODE:

\[
\frac{dS}{dt} = -pS (1 - \exp (-Ia)) ; \quad \frac{dI}{dt} = pS (1 - \exp (-Ia)) - \gamma I ; \quad \frac{dR}{dt} = \gamma I \quad (3)
\]
Estimating parameters from data

\[
\frac{dS}{dt} = -\beta S \left( 1 - \exp \left( -\frac{I}{N} \alpha \right) \right) \frac{1}{\alpha}; \quad \frac{dI}{dt} = \beta S \left( 1 - \exp \left( -\frac{I}{N} \alpha \right) \right) \frac{1}{\alpha} - \gamma I; \quad \frac{dR}{dt} = \gamma I
\]

- We use nonlinear least-squares fit to minimize the error in both \( I \) and \( R \):

\[
\min_{\alpha, \beta, \gamma} \sum_j (I(t_j) - I_j)^2 + (R(t_j) - R_j)^2
\]

where \( S_j, R_j \) is the data from CSSEGIS or COVID19Tracking websites.

- Recovered = recovered+death; infected = total-recovered

- Used python's optimize routine using reasonable initial conditions (it "just works").

- We optimize over all parameters including gamma; but we fit to both \( I \) and \( R \).

- Used data up to June 13, 2020, and from the time of 100 recovered individuals.

- Here are results:
Sometimes it doesn't work:
Predictive value

Training on all of data

Training on half of data, predicting other half
Comparison: saturation vs. without
Effect of population density on $R_0$

- SIR with saturation
  \[
  \frac{dS}{dt} = -pS(1 - \exp(-Ia)) ; \quad \frac{dI}{dt} = pS(1 - \exp(-Ia)) - \gamma I; \quad \frac{dR}{dt} = \gamma I \quad (4)
  \]

- For small $aI$, we linearize $1 - \exp(Ia) \sim aI$ so the model becomes
  \[
  \frac{dS}{dt} = -paSI; \quad \frac{dI}{dt} = paSI - \gamma I; \quad \frac{dR}{dt} = \gamma I \quad (5)
  \]

- Here, $a = \pi r^2/A$, where $A$ is total area and $r$ infection radius.

- In terms of population density, $A = N/\rho$ so we have:
  \[
  a = \pi r^2 \frac{\rho}{N}
  \]

- Compared to the usual SIR model, $pa = \beta/N$ and
  \[
  R_0 = \frac{\beta}{\gamma} = \frac{pNa}{\gamma} = \rho \frac{pr^2\pi}{\gamma}
  \]

  where $\rho$ is the population density.

- Conclusion: $R_0$ is proportional to the population density at the onset of the outbreak.
# SIR classical vs. SIR saturated

<table>
<thead>
<tr>
<th></th>
<th>SIR with saturation</th>
<th>Classical SIR</th>
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</thead>
<tbody>
<tr>
<td><strong>Equations</strong></td>
<td>[ \frac{dS}{dt} = -pS (1 - \exp (-Ia)) ; ] [ \frac{dI}{dt} = pS (1 - \exp (-Ia)) - \gamma I ]</td>
<td>[ \frac{dS}{dt} = -\beta SI / N ; ] [ \frac{dI}{dt} = \beta SI / N - \gamma I ]</td>
</tr>
<tr>
<td><strong>Parameters</strong></td>
<td>[ a = \frac{\rho}{N} r^2 \pi ] includes { pop density ( \rho ), infection radius ( r ) }</td>
<td>[ \beta \equiv \text{infection rate} ] [ \gamma \equiv \text{recovery rate} ]</td>
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<td></td>
<td>( p ) : probability of infection [ \gamma \equiv \text{recovery rate} ]</td>
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<tr>
<td><strong>( R_0 )</strong></td>
<td>[ \rho p\frac{r^2 \pi}{\gamma} ] [ \beta / \gamma ]</td>
<td></td>
</tr>
<tr>
<td><strong>( I_{\text{max}} / N )</strong></td>
<td>( I_{\text{max, sat}} &lt; I_{\text{max, classical}} )</td>
<td>[ 1 + \frac{1}{R_0} \left( \log \frac{1}{R_0} - 1 \right) ]</td>
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<tr>
<td><strong>Attack rate ( r_\infty )</strong></td>
<td>( r_{\infty, sat} &lt; r_{\infty, \text{classical}} )</td>
<td>[ \ln \left( 1 - r_{\infty} \right) / r_{\infty} = R_0 ]</td>
</tr>
</tbody>
</table>
Red = I (classic), blue = I (saturation)

\[ p = 0.01 \times 2^k, \ k = 0..6 \]
\[ a = 0.006/p, \ \text{gamma}=0.2, \ \text{N}=100 \]

\[ a = 0.3, \ p=0.02, \ \text{gamma}=0.05 \times 2^k, \ k = 0..3 \]
Social distancing

• Suppose that each person has a "hard-core" repulsion radius $h$ in addition to "infection radius" $r$.

• Let $E$ be the expected area fraction for $I$ infected individuals. Then

\[
E = I a - \binom{I}{2} \frac{(a - b)^2}{1 - 2b} + \binom{I}{3} \frac{(a - 2b)^3}{(1 - 3b)^2} \cdots
\]

\[
E = \sum_{j=1}^{I} \left(-1\right)^j \binom{I}{j} I_j
\]

where

\[
I_j = \begin{cases} 
\frac{(a - (j - 1)b)^j}{(1 - jb)^{j-1}}, & \text{if } a > (j - 1)b \\
0, & \text{if } a < (j - 1)b 
\end{cases}
\]

\[
a = \frac{\pi r^2}{A}, \quad b = \frac{\pi h^2}{A}
\]

• The probability of coming into contact with one of $I$ infected individuals is then given by:

\[
F(I; a, b) := \frac{E(k, a, b) - Ib}{1 - Ib}
\]
- Note that $F \sim (1 - \exp (-K a))$ when $b = 0$ so this generalizes the $b = 0$ case.

- Note that $F \sim I (a - b)$ for small $I$, which reduces to the usual law of mass action (linear in $I$)

● Conclusion:

Replace $\frac{\beta}{N} SI$ by $\beta F(I, \frac{\alpha}{N}, \frac{\mu}{N})$

$$\frac{dS}{dt} = -\beta SF(I, \frac{\alpha}{N}, \frac{\mu}{N}); \quad \frac{dI}{dt} = \beta SF(I, \frac{\alpha}{N}, \frac{\mu}{N}) - \gamma I; \quad \frac{dR}{dt} = \gamma I$$  (6)
$t = 0$

$\text{Theory for hard-core saturation}$

$\text{Simulations with hard-core}$

$\text{Theory without hard-core}$

$a=0.0201, b=0.0113, p=0.0300$
Future exploration

- Motion of agents (not fully-mixed), spatial propagation

- Piecewise parameter fit: Especially for European countries (e.g. Germany)

- Data fitting with parameters

- Decaying kernels

- Effect of social distancing

Thank you!