Vortex dynamics, animal skin patterns, and ice fishing

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Joint works with Michael Ward and Juncheng Wei, Yuxin Chen, Daniel Zhirov, Ricardo Carretero, Panoyatis Keverkedis
**Vortex dynamics**

- Equations first given by Helmholtz (1858): each vortex generates a rotational velocity field which advects all other vortices. **Vortex model:**

  \[
  \frac{dz_j}{dt} = i \sum_{k \neq j} \gamma_k \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1 \ldots N.
  \]

- Classical problem; observed in many physical experiments: floating magnetized needles (Meyer, 1876); Malmberg-Penning trap (Durkin & Fajans, 2000), Bose-Einstein Condensates (Ketterle et.al. 2001); magnetized rotating disks (Whitesides et.al, 2001)

- Conservative, Hamiltonian system

- General initial conditions lead to chaos: *movie — chaos*

- Certain special configurations are “stable” in Hamiltonian sense: *movie — stable*

- Rigidly rotating steady states are called **relative equilibria:**

  \[
  z_j(t) = e^{\omega t} \xi_j \iff 0 = \sum_{k \neq j} \gamma_k \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} - \omega \xi_j
  \]
Dynamic, self-assembled aggregates of magnetized, millimeter-sized objects rotating at the liquid-air interface: Macroscopic, two-dimensional classical artificial atoms and molecules

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Figure 1 Experimental set-up and magnetic force profiles. (a) A scheme of the experimental set-up. A bar magnet rotates at angular velocity $\omega$ below a dish filled with lid (typically ethylene glycol/water or glycerine/water solutions). Magnetically doped ss are placed on the liquid–air interface, and are fully immersed in the liquid except for r top surface. The disks spin at angular velocity $\omega$ around their axes. A magnetic force attracts the disks towards the centre of the dish, and a hydrodynamic force $F_T$ pushes...
Fig. 1. Observation of vortex lattices. The examples shown contain approximately (A) 16, (B) 32, (C) 80, and (D) 130 vortices. The vortices have "crystallized" in a triangular pattern. The diameter of the cloud in (D) was 1 mm after ballistic expansion, which represents a magnification of 20. Slight asymmetries in the density distribution were due to absorption of the optical pumping light.
Campbell and Ziff (1978) classified many stable configurations for small (e.g. \( N = 18 \)) number of vortices of equal strength.

Goal: describe the stable configuration in the continuum limit of a large number of vortices \( N \) (e.g. \( N = 100, 1000 \ldots \)). These have been observed in several recent experiments: Bose Einstein Condensates, magnetized disks.
Key observation

Vortex model: \[ \frac{dz_j}{dt} = i \sum_{k \neq j} \gamma_k \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1 \ldots N. \tag{V} \]

Relative equilibrium: \[ z_j(t) = e^{\omega it} \xi_j \iff 0 = \sum_{k \neq j} \gamma_k \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} - \omega \xi_j \]

Aggregation model: \[ \frac{dx_j}{dt} = \sum_{k \neq j} \gamma_k \frac{x_j - x_k}{|x_j - x_k|^2} - \omega x_j. \tag{A} \]

- One-to-one correspondence between the steady states \( x_j(t) = \xi_j \) of (A) and the relative equilibrium \( z_j(t) = e^{\omega it} \xi_j \) of (V).

- **Spectral equivalence of (V) and (A):** The equilibrium \( x_j(t) = \xi_j \) is asymptotically stable for the aggregation model (A) if and only if the relative equilibrium \( z_j(t) = e^{\omega it} \xi_j \) is stable (neutrally, in the Hamiltonian sense) for the vortex model (V).

- Aggregation model fully describes relative equilibria and their linear stability in the vortex model.

- Aggregation model is easier to study than the vortex model.
Vortices of equal strength $\gamma_k = \gamma$

$$\frac{dz_j}{dt} = i\gamma \sum_{k \neq j} \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1 \ldots N.$$  

- In the limit $N \to \infty$, the steady state density of (A) is constant inside the ball of radius

$$R_0 = \sqrt{N\gamma/\omega}.$$  

Fig. 1. Stable relative equilibria of $N = 25, 50$ and 200 vortices of equal strength. The dashed line shows the analytical prediction $R_0 = \sqrt{N\gamma/\omega}$ of the swarm radius in the $N \to \infty$ limit (see (6)).
Connection to the biological aggregation model

- [FKH, 2011] Multi-particle interaction model:

\[
\frac{dx_j}{dt} = \frac{1}{N} \sum_{k \neq j} \frac{x_j - x_k}{|x_j - x_k|^2} - x_j, \quad j = 1 \ldots N. \tag{1}
\]


\[
\text{Newtonian repulsion} \quad \text{Linear attraction} \tag{2}
\]

- This is just the first two terms of the ice-fishing problem (no reflection in the boundary)
- This model results in a **constant density swarm.**

- **Newtonian** repulsion, **linear** attraction.
- In the limit \( N \rightarrow \infty \), the density is constant inside a ball of radius 1; zero outside.
Continuum limit

- We define the **density** $\rho$ as

$$\int_D \rho(x) \, dx \approx \frac{\# \text{particles inside domain } D}{N}$$

- The flow is then characterized by density $\rho$ and velocity field $v$:

$$\rho_t + \nabla \cdot (\rho v) = 0; \quad v(x) = \int_{\mathbb{R}^n} \left( \frac{x - y}{|x - y|^2} - x - y \right) \rho(y) \, dy.$$  \hspace{1cm} (3)

- We have

$$v(x) = \int \nabla_x \left( \log |x - y| - \frac{1}{2} |x - y|^2 \right) \rho(y) \, dy$$

$$\nabla \cdot v = \int (2\pi \delta(x - y) - 2) \rho(y) \, dy$$

$$= 2\pi \rho(x) - 2M$$
• Inside, the swarm, $\nabla \cdot v = 0 \implies \rho = M/\pi$ is constant!

• Radius is determined by conservation of mass: $M = \rho \pi R^2 \implies R = 1$. 
$N + 1$ problem

- $N$ vortices of equal strength and a single vortex of a much higher strength:

$$\frac{dx_j}{dt} = \frac{a}{N} \sum_{k=1 \ldots N, k \neq j} \frac{x_j - x_k}{|x_j - x_k|^2} + b \frac{x_j - \eta}{|x_j - \eta|^2} - x_j, \quad j = 1 \ldots N, \quad (4)$$

$$\frac{d\eta}{dt} = \frac{a}{N} \sum_{k=1 \ldots N} \frac{\eta - x_k}{|\eta - x_k|^2} - \eta \quad (5)$$

- Mean-field limit $N \to \infty$:

$$\begin{cases} 
\rho_t + \nabla \cdot (\rho \nabla v) = 0; \\
v(x) = a \int_{\mathbb{R}^2} \rho(y) \frac{x-y}{|x-y|^2} dy + b \frac{x-\eta}{|x-\eta|^2} - x. \\
\frac{d\eta}{dt} = a \int_{\mathbb{R}^2} \rho(y) \frac{\eta-y}{|\eta-y|^2} dy - \eta \end{cases} \quad (6)$$

- Main result:. Define $R_1 = \sqrt{b}$, $R_0 = \sqrt{a + b}$ and suppose that $\eta$ is any point such that $B_{R_1}(\eta) \subset B_{R_0}(0)$. Then the equilibrium solution for (6) is constant inside $B_{R_0}(0) \setminus B_{R_1}(\eta)$ and is zero outside.
Unlike the $N+0$ problem, the relative equilibrium for the $N+1$ problem is non-unique: any choice of $\eta$ yields a steady state as long as $|\eta| < R_0 - R_1$. 
Degenerate case: big central vortex

- Small vortices are constrained to a ring of radius $R_0$. with big vortex at the center.

- **Non-uniform** distribution of small particles!

- Question: Determine the size of the gap $\Theta_{gap}$. 
Main Result:

\[ \Theta_{gap} \sim C N^{-1/3}. \]

where the constant \( C = 8.244 \) satisfies

\[
(8 - 6u + 2u^3) \ln(u - 1) = 3u(u^2 - 4); \quad C = 2 \left( \frac{6\pi(2 - u)}{u(u^2 - 1)} \right)^{1/3}
\]
Sketch of proof

- [Barry+Wayne, 2012]: Set \( x_j(t) \sim R_0 e^{i\theta_j(t)} \) then at leading order we get

\[
\frac{d\theta_j}{dt} = \frac{1}{N} \sum_{k \neq j} \left( \frac{\sin(\theta_j - \theta_k)}{2 - 2 \cos(\theta_j - \theta_k)} - \sin(\theta_j - \theta_k) \right).
\]  
(7)

- In the mean-field limit \( N \to \infty \), the density distribution \( \rho(\theta) \) for the angles \( \theta_j \) satisfies

\[
\begin{cases}
\rho_t + (\rho v_\theta)_\theta = 0, \\
v(\theta) = PV \int_{-\pi}^{\pi} \rho(\phi) \left( \frac{\sin(\theta - \phi)}{2 - 2 \cos(\theta - \phi)} - \sin(\theta - \phi) \right) d\phi,
\end{cases}
\]  
(8)

where \( PV \) denotes the principal value integral, and \( \int_{-\pi}^{\pi} \rho = 1 \).

- [Barry, PhD Thesis]: Up to rotations, the steady state density \( \rho(\theta) \) for which \( v = 0 \) must be of the form

\[
\rho(\theta) = \frac{1}{2\pi} \left( 1 + \alpha \cos \theta \right).
\]  
(9)

This follows from (8) and (formal) expansion

\[
\frac{\sin t}{2 - 2 \cos t} - \sin t = \sin(2t) + \sin(3t) + \sin(4t) + \ldots
\]
• $\alpha$ is free parameter in the continuum limit.

• For discrete $N$, particle positions satisfy

$$\int_{\theta_{j-1}}^{\theta_j} \frac{1}{2\pi} (1 + \alpha \cos \theta) \, d\theta = \frac{1}{N}$$

To estimate $\Phi_{gap}$, choose $\theta_1$ so that $v(\theta_1) \sim 0$. See our paper for hairy details.
**N + K problem**

\[ v(x) = a \int_{\mathbb{R}^2} \rho(y) \frac{x - y}{|x - y|^2} dy + \sum_{k=1}^{K} b_k \frac{x - \eta_k}{|x - \eta_k|^2} - x, \]

\[ \frac{d\eta_j}{dt} = a \int_{\mathbb{R}^2} \rho(y) \frac{\eta_k - y}{|\eta_k - y|^2} dy + \sum_{\substack{k=1 \ldots K \atop k \neq j}} b_k \frac{\eta_j - \eta_k}{|\eta_j - \eta_k|^2} - \eta_j, \]

\[ j = 1 \ldots K. \]

**Main result:** Let \( R_k = \sqrt{b_k}, \ k = 1 \ldots K \) and \( R_0 = \sqrt{a + b_1 + \ldots + b_K} \). Suppose \( \eta_1 \ldots \eta_K \) are such \( B_{R_1}(\eta_1) \ldots B_{R_K}(\eta_K) \) are all disjoint and are contained inside \( B_{R_0}(0) \). The equilibrium density is constant inside \( B_{R_0}(0) \setminus \bigcup_{k=1}^{K} B_{R_k}(\eta_k) \) and is zero outside.
\( N + K \) problem, with very large \( K \) vortices

- The \textbf{blue ellipse} is described by the reduced system

\[
\frac{d\xi_j}{dt} = \frac{1}{N} \sum_{k=1, k \neq j}^{N} \frac{1}{\xi_j - \xi_k} + \frac{1}{2} \xi_k - \xi_k
\]

- From [K, Huang, Fetecau, 2001], its axis ratio is 3.
Crystallization

Vortex model: \[ \frac{dz_j}{dt} = i \sum_{k \neq j} \gamma_k \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1 \ldots N. \quad (V) \]

Relative equilibria: \[ z_j(t) = e^{\omega it} \xi_j \iff 0 = \sum_{k \neq j} \gamma_k \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} - \omega \xi_j \]

Vortex with dissipation: \[ \frac{dz_j}{dt} = i \sum_{k \neq j} \gamma_k \frac{z_j - z_k}{|z_j - z_k|^2} + \mu \left( \sum_{k \neq j} \gamma_k \frac{z_j - z_k}{|z_j - z_k|^2} - \omega z_j \right) \quad (D) \]

• In many physical experiments of BEC there is damping or dissipation involved.

• **Spectral equivalence:** Relative equilibria and their stability are the same for (V) and (D).

• Both the vortex model and the “aggregation model” model are limiting cases of (D).

• Taking \( \mu > 0 \) stabilizes vortex dynamics! chaos damped stable

• This allows us to find stable relative equilibria numerically.
Vortex dynamics in BEC with trap

- For BEC, dynamics have extra term corresponding to precession around the trap:

\[ \dot{z}_j = i \frac{a}{1 - r^2} z_j + iC \sum_{k \neq j} \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1 \ldots N. \tag{11} \]

\[ \text{trap-interaction} \quad \text{self-interaction} \]

- Large \( N \) limit: \textbf{non-uniform} vortex lattice:

\[
\rho \sim \omega - \frac{a}{(1 - r^2)^2} \quad \text{if} \quad r < R, \quad \rho = 0 \quad \text{otherwise},
\]

with \( \omega = \frac{a}{1 - R^2} + \frac{cN}{R^2} \)
\[ \omega_c = \left( \sqrt{a} + \sqrt{cN} \right)^2; \quad R_c^2 = \frac{\sqrt{cN}}{\sqrt{a} + \sqrt{cN}}. \]

- No solutions of \( \omega < \omega_c \)
- Two solutions \( R = R_\pm \) if \( \omega > \omega_c \), smaller is stable, larger unstable.
N-body problem

\[ \dot{z}_j = \sum_{k \neq j} c_k c_j \frac{z_k - z_j}{|z_k - z_j|^3} \]  

(12)

- Relative equilibria \( z_j = e^{i\omega t} x_j \) satisfy:

\[ 0 = \sum_{k \neq j} c_k c_j \frac{x_k - x_j}{|x_k - x_j|^3} + \omega^2 x_j \]  

(13)

- Gradient flow (to find steady states):

\[ -\dot{x}_j = \sum_{k \neq j} c_k c_j \frac{x_k - x_j}{|x_k - x_j|^3} + \omega^2 x_j \]  

(14)
• For $N$ equal-mass bodies, the relative equilibrium is known to be unstable when $N \geq 3$.

• Unlike the vortex model, there is no spectral equivalence between (12) and (14)
Spot solutions in Reaction-diffusion systems

seashells * fish * crime hotspots in LA * stressed bacterial colony
Classical Gierer-Meinhardt model

\[ A_t = \varepsilon^2 \Delta A - A + \frac{A^2}{H}; \quad \tau H_t = D \Delta H - H + A^2 \]

- Introduced in 1970's to model cell differentiation in hydra
- Mostly of mathematical interest: one of the simplest RD systems
- Has been intensively studied since 1990's [by mathematicians!]
- Key assumption: separation of scales

\[ \varepsilon \ll 1 \text{ and } \varepsilon^2 \ll D. \]
• Roughly speaking, $H$ is constant on the scale of $A$ so the steady state looks "roughly" like $A(x) \sim Cw \left( \frac{x - x_0}{\varepsilon} \right)$ where

$$\Delta w - w + w^2 = 0.$$ 

• Questions: What about stability? What about location of the spike $x_0$?
“Classical” Results in 1D:

• Wei 97, 99, Iron+Wei+Ward 2000: Stability of $K$ spikes in the GM model in one dimension

• Two types of possible instabilities: structural instabilities or translational instabilities

• Structural instabilities (large eigenvalues) lead to spike collapse in $O(1)$ time

• Translational instabilities can lead to ”slow death”: spikes drift over large time scales

• **Main result 1**: There exists a sequence of thresholds $D_K$ such that $K$ spikes are stable iff $D < D_K$.

• **Main result 2**: Slow dynamics of $K$ spikes is described by an ODE with $2K$ variables (spike heights and centers) subject to $K$ algebraic constraints between these variables.
Large eigenvalues

• Careful derivation leads to a nonlocal eigenvalue problem (NLEP) of the form

\[
\lambda \phi = \Delta \phi + (-1 + 2w) \phi - \chi w^2 \frac{\int w \phi}{\int w^2}; \quad \chi := \frac{4 \sinh^2 \left( \frac{1}{\sqrt{D}} \right)}{2 \sinh^2 \left( \frac{1}{\sqrt{D}} \right) + 1 - \cos \left[ \pi (1 - 1/K) \right]}
\]

• Key theorem (Wei, 99): \( \text{Re}(\lambda) < 0 \) iff \( \chi < 1 \)

• Corollary: On a domain \([-1, 1]\), large eigenvalues are stable iff \( D < D_{K,\text{large}} \) where

\[
D_{K,\text{large}} = \frac{1}{\arcsinh^2 (\sin 2\pi / K)}
\]

• When unstable, this can lead to competition instability.

• Movies: stable; unstable
Small eigenvalues

- Causes a very slow drift

- Iron-Ward-Wei 2000: The slow dynamics of the system can be reduced to a coupled algebraic-differential system of ODEs

- Movie: slow drift
Two dimensions

- Structural stability is similar


\[
\frac{dx_0}{dt} \sim -\frac{4\pi \varepsilon^2}{\ln \varepsilon^{-1} + 2\pi R_0} \nabla R_0
\]

where

\[
R_0 = \lim_{x \to x_0} \left[ G(x, x_0) + \frac{1}{2\pi} \ln(|x - x_0|) \right];
\]

\[
\nabla R_0 = \lim_{x \to x_0} \nabla_x \left[ G(x, x_0) + \frac{1}{2\pi} \ln(|x - x_0|) \right];
\]

\[
\Delta G - \frac{1}{D} G = -\delta (x - x_0) \text{ on } \Omega; \quad \partial_n G = 0 \text{ on } \partial\Omega
\]

- Equilibrium location \( x_0 \) satisfies \( \nabla R_0 = 0 \), occurs at the extremum of the regular part of the Neumann's Green's function
Dumbbell-shaped domain

- QUESTION: Suppose that a domain has a dumb-bell shape. Where will the spike drift??

- What are the possible equilibrium locations for a single spike?
Small $D$ limit

- If $D$ is very small, $R_0(x_0) \sim C(x_0) \exp \left(-\frac{1}{\sqrt{D}} |x_0 - x_m| \right)$ where $x_m$ is the point on the boundary closest to $x_0$

- This means that $R_0$ is minimized at the point furthest away from the boundary when $D \ll 1$
  - In the limit $\varepsilon^2 \ll D \ll 1$, the spike drifts towards the point furthest away from the boundary.
  - For a dumbbell-shaped domain above, the three possible equilibria are at the "centers" of the dumbbells (stable) and at the center of the neck (unstable saddle point)
  - For multiple spikes, their locations solve "ball-packing problem".

- Movie: $D = 0.03, \varepsilon = 0.04$
Large D limit

- We get the modified Green’s function:
  \[
  \Delta G_m - \frac{1}{|\Omega|} = -\delta(x - x_0) \text{ inside } \Omega, \quad \partial_n G = 0 \text{ on } \partial\Omega;
  \]

  \[
  R_{m0} = \lim_{x \to x_0} \left[ G_m(x, x_0) + \frac{1}{2\pi} \ln(|x - x_0|) \right].
  \]

- [K, Ward, 2003]: For a domain which is an analytic mapping of a unit disk, \( \Omega = f(B) \), we derive an exact formula for \( \nabla R_{m0} \) in terms of the residues of \( f(z) \) outside the unit disk.

- Take \( f(z) = \frac{(1 - a^2)z}{z^2 + a^2} \); \( x_0 = f(z_0) \):
Then

\[ \nabla R_{m0}(x_0) = \frac{\nabla s(z_0)}{f'(z_0)} \]

where

\[
\nabla s(z_0) = \frac{1}{2\pi} \left( \frac{z_0}{1 - |z_0|^2} - \frac{(\bar{z}_0^2 + 3a^2)\bar{z}_0}{\bar{z}_0^4 - a^4} + \frac{a^2\bar{z}_0}{\bar{z}_0^2a^2 - 1} + \frac{\bar{z}_0}{\bar{z}_0^2 - a^2} \right)
\]

• Corollary: for above Ω, \( \nabla R_{m0} \) has a unique root at the origin!

  - In the limit \( D \gg 1 \), all spikes will drift towards the neck.

• Complex bifurcation diagram as \( D \) is increased.

• Movie: \( \varepsilon = 0.05, \; D = 0.1; \; D = 1. \)
"Huge" $D$

- In the limit $D \to \infty$, (Shadow limit), an interior spike is unstable and moves towards the boundary [Iron Ward 2000; Ni, Poláčik, Yanagida, 2001].

- For \textbf{exponentially large but finite} $D = O(\exp(-C/\varepsilon))$, boundary effects will compete with the Green's function.

- [K, Ward, 2004]: Define

\[
\sigma := \frac{\varepsilon}{2} \ln \left( \frac{C_0}{|\Omega|} D \varepsilon^{-1/2} \right); \quad C_0 \approx 334.80;
\]

Then the spike will move towards the boundary whenever its distance from the closest point of the boundary is at most $\sigma$; otherwise it will move away from the boundary.

- \textbf{Movies}: $\varepsilon = 0.05$, $D = 10$; $D = 100$
Spike dynamics inside a disk

In the limit $\varepsilon \ll 1, D \gg 1$, inside the disk we get

$$C \frac{dx_j}{dt} \sim 2 \sum_{k \neq j} \frac{x_j - x_k}{|x_j - x_k|^2} - \sum_k x_j + \sum_k \frac{x_j - x_k/|x_k|^2}{|x_j - x_k/|x_k|^2|^2} - \sum_k -\frac{x_j |x_k|^2 + x_k |x_j|^2}{|x_j |x_k|^2 - x_k^2}. $$

The first two terms are identical to vortex stability model!

The last two terms represent “reflection in the wall”

Just like for vortex model, the steady state consists of uniformly-distributed particles inside the domain!

Movies: disk; dumbbell.
Mean first passage time (ice fishing)

- Question: Suppose you want to catch a fish in a lake covered by ice. Where do you drill a hole to maximize your chances?

- Related questions: cell signalling; oxygen transport in muscle tissues; cooling rods in a nuclear reactor...

- Consider $N$ non-overlapping small "holes" each of small radius $\varepsilon$. A particle is performing a random walk inside the domain $\Omega$. If it hits a hole, it gets destroyed; if it hits a boundary, it gets reflected. Question: what is the expected lifetime of the wondering particle? How do we place the holes to minimize this lifetime [i.e. catch the fish, cool the nuclear reactor...]?
The expected lifetime is proportional to $1/\lambda$ where $\lambda$ is the smallest eigenvalue of the problem:

$$\Delta u + \lambda u = 0 \text{ inside } \Omega \setminus \Omega_p; \quad u = 0 \text{ on } \partial \Omega_p; \quad \partial_n u = 0 \text{ on } \partial \Omega$$

where $\Omega_p = \bigcup_{i=1}^{N} \Omega_\varepsilon$.

[K-Ward-Titcombe, 2005]: The smallest eigenvalue is given by

$$\lambda \sim \frac{2\pi N}{\ln \frac{1}{\varepsilon}} \left( 1 - \frac{2\pi}{\ln \frac{1}{\varepsilon}} p(x_1, \ldots x_N) + O \left( \frac{1}{(\ln \frac{1}{\varepsilon})^2} \right) \right)$$

where

$$p(x_1, \ldots x_N) := \sum \sum G_{ij};$$

$$G_{ij} = \begin{cases} G_m(x_i, x_j) & \text{if } i \neq j \\ R_m(x_i, x_i) & \text{if } i = j \end{cases}$$

$$\Delta G_m(x, x') - \frac{1}{|\Omega|} = -\delta(x - x') \text{ inside } \Omega, \quad \partial_n G = 0 \text{ on } \partial \Omega; \quad R_m \equiv \text{reg.part}$$

For a unit disk:

$$2\pi G_m(x, x') = -\ln |x - x'| - \ln \left| x \frac{x'}{|x'|} \right| - \frac{x'}{|x'|} + \frac{1}{2} \left( |x|^2 + |x'|^2 \right)$$

$$2\pi R_m(x, x') = -\ln \left| x \frac{x'}{|x'|} - \frac{x'}{|x'|} \right| + \frac{1}{2} \left( |x|^2 + |x'|^2 \right)$$

The optimum trap placement is at the minimum of $p(x_1, \ldots x_N)$.
Disk domain, $N$ holes

We need to minimize

$$p(x_1 \ldots x_N) = -\sum_{j \neq k} \ln |x_j - x_k| - \sum_{j,k} \left( \ln \left| x_j - \frac{x_k}{|x_k|^2} \right| + \ln |x_k| \right) + \frac{1}{2} \sum_{j,k} \left( |x_j|^2 + |x_k|^2 \right)$$

Gradient flow is uniform swarm model plus two extra terms

$$\frac{dx_j}{dt} = 2 \sum_{k \neq j} \frac{x_j - x_k}{|x_j - x_k|^2} - \sum_k x_j + \sum_k \frac{x_j - x_k/|x_k|^2}{|x_j - x_k/|x_k|^2|^2} - \sum_k \frac{-x_j |x_k|^2 + x_k |x_j|^2}{|x_k|^2 - x_j|^2}.$$ 

Particles on a ring: $x_k = r e^{i k 2\pi / N}$. The min occurs when

$$\frac{r^{2N}}{1 - r^{2N}} = \frac{N - 1}{2N} - r^2$$

Note that $r \to 1/\sqrt{2}$ as $N \to \infty$; the optimal ring divides the unit disk into two equal areas.

Particles on 2,3,... $m$ rings: Similar results are derived with complicated but numerically useful formulas.
Constrained optimization on up to 3 rings
Full optimization of $K$ traps

6 (−1.526)  7 (−1.8871)  8 (−2.2538)  9 (−2.6104)  10 (−2.976)  
11 (−3.3562)  12 (−3.7593)  13 (−4.1552)  14 (−4.5683)  15 (−4.975)  
16 (−5.3914)  17 (−5.8051)  18 (−6.2245)  19 (−6.6731)  20 (−7.1071)  
21 (−7.5489)  22 (−7.985)  23 (−8.4207)  24 (−8.8693)  25 (−9.3178)
Comparison

10, -2.96861, -2.976

15, -4.97285, -4.97502

13, -4.1511, -4.15515

24, -8.85623, -8.86797
Conclusion

• We looked at three very different problems: vortex dynamics; spike dynamics and first mean-passage time

• All three problems reduce to nonlocal particle aggregation model with Newtonian repulsion

• In the limit of large number of particles, the steady state approaches a uniform distribution.

• Spectral equivalence of aggregation and vortex model shows stability

These papers are available for download from my website: http://www.mathstat.dal.ca/~tkolokol

Thank you! Any questions?