Predator-swarm interactions and related topics

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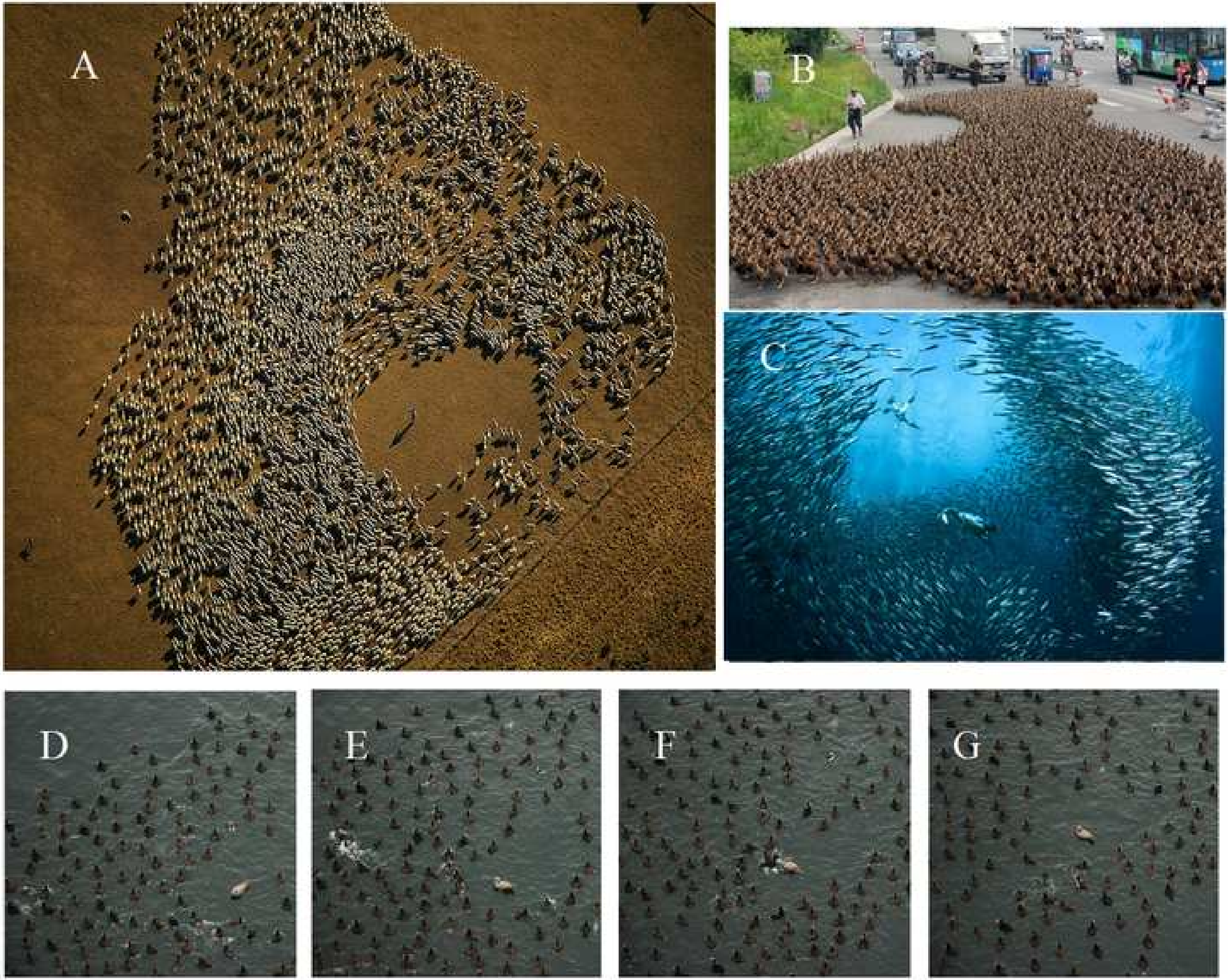
Joint works with Yuxin Chen, Daniel Zhirov, Ricardo Carretero and Panoyatis Keverkedis

Predator-swarm interactions

- Collective behaviour occur at all levels of living organisms, from bacterial colonies to fish schools to to human cities.
- Hypothesis: swarming behaviour is an evolutionary adaptation that confers certain benefits on the individuals or group as a whole [Parrish,Edelstein-Keshet 1999; Sumpter 2010, Krause&Ruxton2002, Penzhorn 1984]
- Benifits:
  - efficient food gathering [Traniello1989]
  - heat preservation in penguins huddles [Waters,Blanchette&Kim 2012]
  - ***predator avoidance*** in fish shoals [Pitcher&Wyche 83] or zebra [Penzhorn84]
    - evasive maneuvers,
    - confusing the predator,
    - safety in numbers
    - increased vigilance
- Counter-hypothesis: swarming can also be detrimental to prey
  - Makes it easier for the predator to spot and attack the group as a whole [Parrish,Edelstein-Keshet 1999].

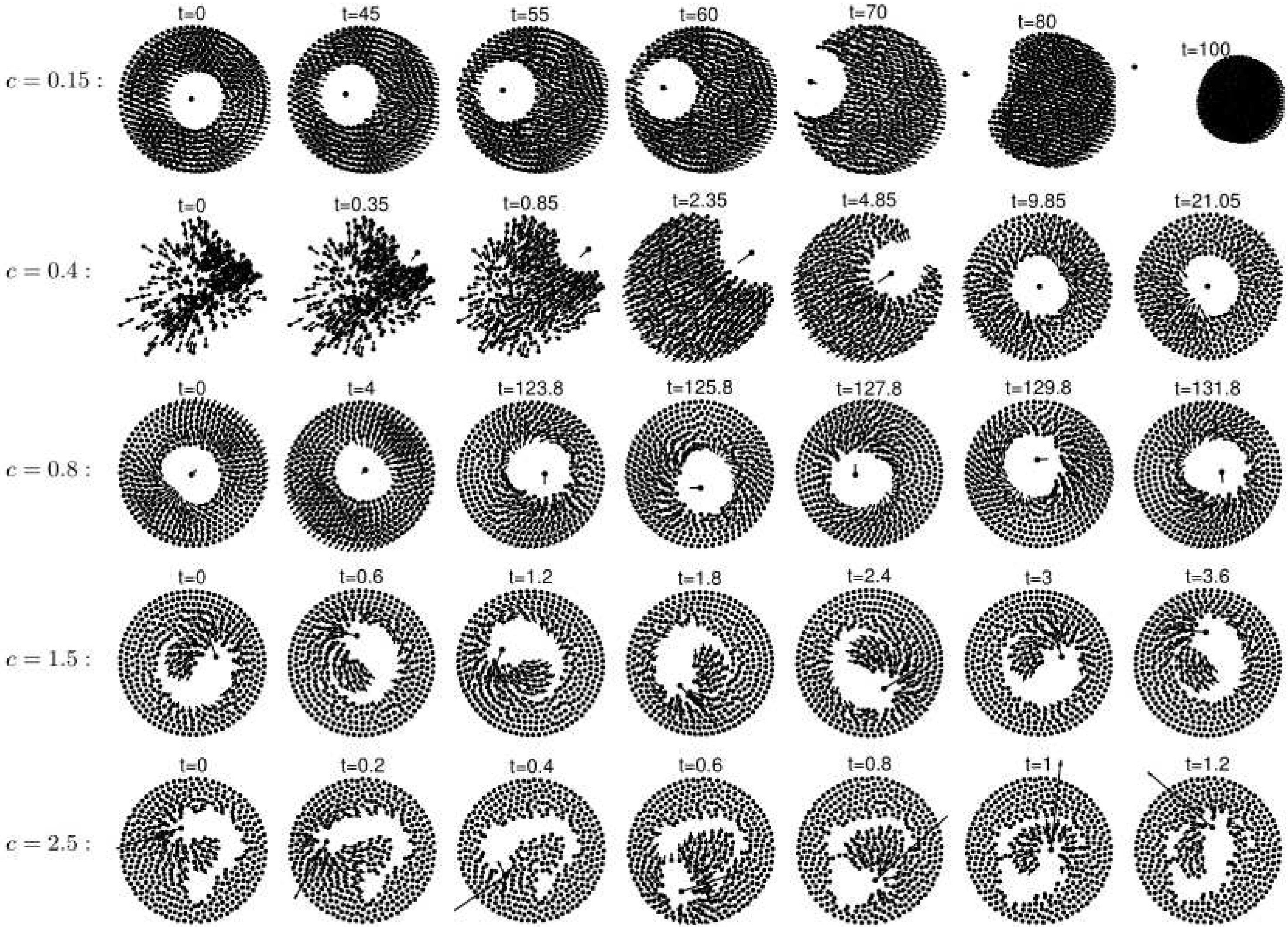

A
B
C
D
E
F
G

Minimal model of predator-swarm interaction

$$\underbrace{\frac{dx_j}{dt}}_{\text{Prey}} = \frac{1}{N}\sum_{k=1,k\neq j}^{N}\left(\underbrace{\frac{x_j - x_k}{|x_j - x_k|^2}}_{\text{prey-prey repulsion}} - \underbrace{a\left(x_j - x_k\right)}_{\text{prey-prey attraction}}\right) + b\underbrace{\frac{x_j - z}{|x_j - z|^2}}_{\text{prey-predator repulsion}} \tag{1}$$

$$\underbrace{\frac{dz}{dt}}_{\text{Predator}} = \frac{c}{N}\sum_{k=1}^{N}\underbrace{\frac{x_k - z}{|x_k - z|^p}}_{\text{predator-prey attraction}}. \tag{2}$$

- We take prey-prey and prey-redator interactions to be ***Newtonian***
  - makes the analysis possible!
- c : predator "strength". We will use it as control parameter.
- p : predator "sensitivity".


$c = 0.15$:
t=0
t=45
t=55
t=60
t=70
t=80
t=100
$c = 0.4$:
t=0
t=0.35
t=0.85
t=2.35
t=4.85
t=9.85
t=21.05
$c = 0.8$:
t=0
t=4
t=123.8
t=125.8
t=127.8
t=129.8
t=131.8
$c = 1.5$:
t=0
t=0.6
t=1.2
t=1.8
t=2.4
t=3
t=3.6
$c = 2.5$:
t=0
t=0.2
t=0.4
t=0.6
t=0.8
t=1
t=1.2

Continuum limit

Coarse grain:

$$\rho\,(x) = \frac{1}{N}\sum_{j=1}^{N}\delta(x - x_j)$$

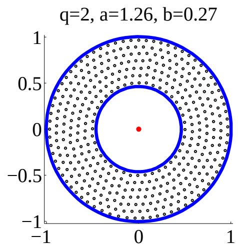
Let $N \to \infty$ we get

$$\rho_t(x,t) + \nabla\cdot(\rho(x,t)v(x,t)) = 0 \tag{3}$$

$$v(x,t) = \int_{\mathbb{R}^2}\left(\frac{x-y}{|x-y|^2} - a\,(x-y)\right)\rho(y,t)dy + b\frac{x-z}{|x-z|^2} \tag{4}$$

$$\frac{dz}{dt} = c\int_{\mathbb{R}^2}\frac{y-z}{|y-z|^p}\rho(y,t)dy. \tag{5}$$

Ring state ("confused" predator)



- Define
$$R_1 = \sqrt{b/a};\ R_2 = \sqrt{(1+b)/a}. \tag{6}$$
The system (3-5) admits a steady state for which $z = 0$, ρ is a positive constant inside an annulus $R_1 < |x| < R_2$, and is otherwise.

- ***Main result of the paper:*** The ring is stable whenever $2 < p < 4$ and
$$\frac{ba^{\frac{2-p}{2}}}{(1+b)^{\frac{2-p}{2}}} =: c_0 < c < c_{hopf} := \frac{a^{\frac{2-p}{2}}}{b^{\frac{2-p}{2}} - (1+b)^{\frac{2-p}{2}}}. \tag{7}$$

- Increasing c past c_{hopf} ***triggers hopf bifurcation!***

Key calculation 1

Define characterisitic coordinates:

$$\frac{dX}{dt} = v(X,t); \quad X(X_0, 0) = X_0. \tag{8}$$

Recall:

$$v(x,t) = \int_{\mathbb{R}^2} \Bigg(\underbrace{\frac{x-y}{|x-y|^2}}_{\nabla_x \ln|x-y|} - a\,(x-y) \Bigg) \rho(y,t)dy + b \Bigg(\underbrace{\frac{x-z}{|x-z|^2}}_{\nabla_x \ln|x-z|} \Bigg)$$

$$\begin{aligned} \nabla_x \cdot v &= \int_{\mathbb{R}^2} [2\pi\delta(x-y) - 2a]\rho(y)dy + 2\pi b\delta(x-z) \\ &= 2\pi\rho(x) - 2aM \end{aligned}$$

So along characteristics,

$$\frac{d\rho}{dt} = -\left(\nabla_x \cdot v\right)\rho \tag{9}$$

$$(2aM - 2\pi\rho)\rho \tag{10}$$

- Conclusion 1: $\rho \to aM/\pi$ as $t \to \infty$
  - $\rho \to \mathrm{const}$ ***regardless of the swarm shape!***
- Conclusion 2: Radial steady state is an annulus of constant density whose dimensions are as above.

Key calculation 2

- The density quickly approaches a constant, so the swarm is fully characterised by the motion of its boundaries.

- To determine its stability, it's enough perturb the boundary and the predator at the center:

$$\text{Inner boundary: } x = R_1 e^{i\theta} + \varepsilon_1 e^{\lambda t} \tag{11}$$

$$\text{Outer boundary: } x = R_2 e^{i\theta} + \varepsilon_2 e^{\lambda t} \tag{12}$$

$$\text{Predator: } z = 0 + \varepsilon_3 e^{\lambda t} \tag{13}$$

- Get a 3x3 eigenvalue problem

$$\left(R_2^2 - R_1^2\right) \lambda \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = A \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} \tag{14}$$

where

$$A = \begin{pmatrix} -b-1 & b & 1 \\ -b - \frac{b}{1+b} & b & \frac{b}{1+b} \\ -c\left(\frac{b}{a}\right)^{\frac{2-p}{2}} & c\left(\frac{1+b}{a}\right)^{\frac{2-p}{2}} & c\left[\left(\frac{b}{a}\right)^{\frac{2-p}{2}} - \left(\frac{1+b}{a}\right)^{\frac{2-p}{2}}\right] \end{pmatrix}.$$

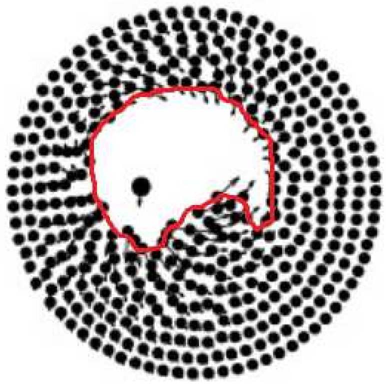
Implications

$$c_{hopf} = \frac{a^{\frac{2-p}{2}}}{b^{\frac{2-p}{2}} - (1+b)^{\frac{2-p}{2}}}, \quad 2 < p < 4 \tag{15}$$

- c_{hopf} is an increasing function of b (prey-predator repulsion)
  - increasing b makes it harder for the predator to catch the prey.
- c_{hopf} is a decreasing function of a (prey-prey attraction strength)
  - increasing a makes it easier for the predator to catch the prey.
  - Swarming behaviour makes it ***easier*** for predator to catch prey (i.e. swarming is bad for prey)!
  - Example: in [Fertl&Wursig95] the authors observed groups of about 20-30 dolphins surrounding a school of fish and blowing bubbles underneath it in an apparent effort to keep the school from dispersing, while other members of the dolphin group swam through the resulting ball of fish to feed.

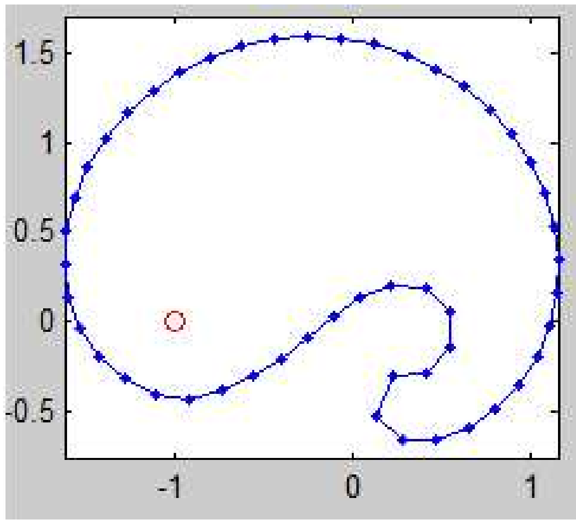
- Swarming may be result of other factors such as food gathering, ease of mating, energetic benifits, or even constraints of physical environment are responsible for prey aggregation.

- When c crosses c_{hopf}, chasing dynamics result. But the prey may still escape!
  - Linear stability is a precursor to capturing the prey, but is insufficient to explain the capturing process itself!
  - Further (non-linear) analysis is needed to explain prey capture.
  - Chasing dynamics "look similar" to shephard chasing sheep:



Far from the ring state

- Transition from oscillatory to chaotic dynamics
- Development of a ”tail” behind the predator
- Predator can catch prey for sufficiently large c.
- Difficult to say anything analytically
  - But can can compute rotating states numerically by evolving the boundary:



Vortex dynamics

- Equations first given by Helmholtz (1858): each vortex generates a rotational velocity field which advects all other vortices. ***Vortex model:***

$$\frac{dz_j}{dt} = i\sum_{k\neq j}\gamma_k \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1\ldots N.$$

- Classical problem; observed in many physical experiments: floating magnetized needles (Meyer, 1876); Malmberg-Penning trap (Durkin & Fajans, 2000), Bose-Einstein Condensates (Ketterle et.al. 2001); magnetized rotating disks (Whitesides et.al, 2001)

- Conservative, hamiltonian system

- General initial conditions lead to chaos: $movie - chaos$

- Certain special configurations are “stable” in hamiltonial sense: $movie - stable$

- Rigidly rotating steady states are called ***relative equilibria***:

$$z_j(t) = e^{\omega i t}\xi_j \quad \Longleftrightarrow \quad 0 = \sum_{k\neq j}\gamma_k \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} - \omega\xi_j$$

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Dynamic, self-assembled aggregates of magnetized, millimeter-sized objects rotating at the liquid-air interface: Macroscopic, two-dimensional classical artificial atoms and molecules

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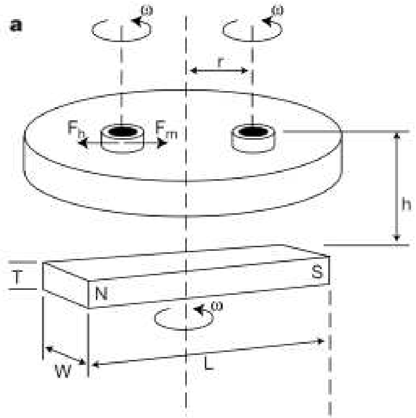

ure 1 Experimental set-up and magnetic force profiles. **a**, A scheme of the erimental set-up. A bar magnet rotates at angular velocity ω below a dish filled with id (typically ethylene glycol/water or glycerine/water solutions). Magnetically doped s are placed on the liquid–air interface, and are fully immersed in the liquid except for ir top surface. The disks spin at angular velocity ω around their axes. A magnetic force attracts the disks towards the centre of the dish, and a hydrodynamic force F_h pushes

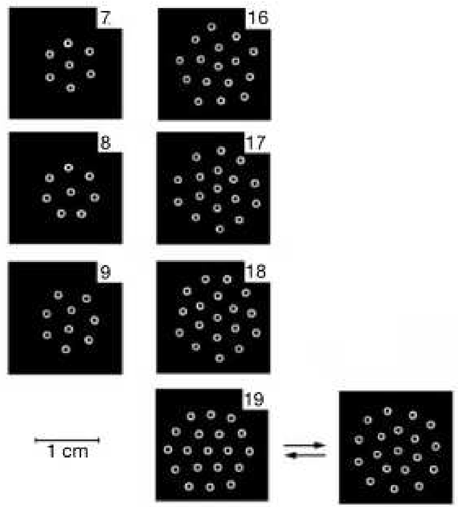

Figure 2 Dynamic patterns formed by various numbers (n) of disks rotating at the ethylene glycol/water–air interface. This interface is 27 mm above the plane of the external magnet. The disks are composed of a section of polyethylene tube (white) of outer diameter 1.27 mm, filled with poly(dimethylsiloxane), PDMS, doped with 25 wt% of magnetite (black centre). All disks spin around their centres at ω = 700 r.p.m., and the entire aggregate slowly (Ω < 2 r.p.m.) precesses around its centre. For $n < 5$, the aggregates do not have a 'nucleus'—all disks are precessing on the rim of a circle. For $n > 5$, nucleated structures appear. For $n = 10$ and $n = 12$, the patterns are bistable in the sense that the two observed patterns interconvert irregularly with time. For $n = 19$, the hexagonal pattern (left) appears only above $\omega \approx 800$ r.p.m., but can be 'annealed' down

Observation of Vortex Lattices in Bose-Einstein Condensates

20 APRIL 2001 VOL 292 SCIENCE

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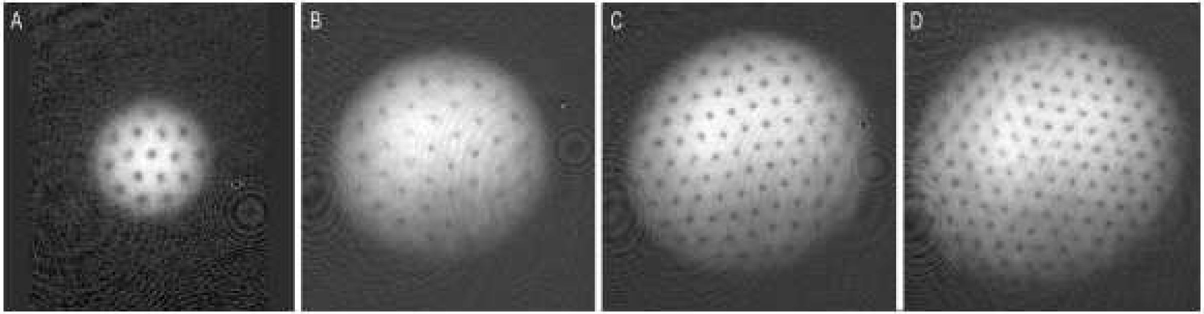
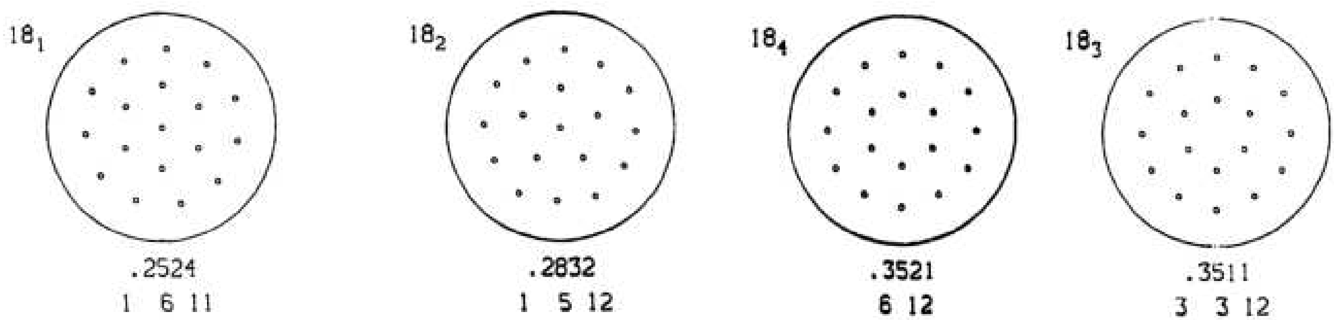


Fig. 1. Observation of vortex lattices. The examples shown contain approximately (**A**) 16, (**B**) 32, (**C**) 80, and (**D**) 130 vortices. The vortices have "crystallized" in a triangular pattern. The diameter of the cloud in (D) was 1 mm after ballistic expansion, which represents a magnification of 20. Slight asymmetries in the density distribution were due to absorption of the optical pumping light.

- Campbell and Ziff (1978) classified many stable configurations for ***small*** $(eg.\ N = 18)$ number of vortices of equal strength.



- Goal: describe the stable configuration in the continuum limit of a ***large*** number of vortices N (eg. $N = 100,\ 1000\ldots$). These have been observed in several recent expriments: Bose Einstein Condensates, magnetized disks

Key observation

$$\text{Vortex model: } \frac{dz_j}{dt} = i\sum_{k\neq j}\gamma_k \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1\dots N. \tag{V}$$

$$\text{Relative equilibrium: } z_j(t) = e^{\omega i t}\xi_j \quad \Longleftrightarrow \quad 0 = \sum_{k\neq j}\gamma_k \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} - \omega\xi_j$$

$$\text{Aggregation model: } \frac{dx_j}{dt} = \sum_{k\neq j}\gamma_k \frac{x_j - x_k}{|x_j - x_k|^2} - \omega x_j. \tag{A}$$

- One-to-one correspondence between the steady statates $x_j(t) = \xi_j$ of (A) and the relative equilibrium $z_j(t) = e^{\omega i t}\xi_j$ of (V).
- ***Spectral equivalence of (V) and (A):*** The equilibrium $x_j(t) = \xi_j$ is asymptotically stable for the aggregation model (A) if and only if the relative equilibrium $z_j(t) = e^{\omega i t}\xi_j$ is stable (neutrally, in the Hamiltonian sense) for the vortex model (V)!
- Aggregation model fully describes relative equilibria and their linear stability in the vortex model.
- Aggregation model is easier to study than the vortex model.

Vortices of equal strength $\gamma_k = \gamma$

Corresponding aggregation model:

$$\frac{dx_j}{dt} = \sum_{k \neq j} \gamma \frac{x_j - x_k}{|x_j - x_k|^2} - \omega x_j. \tag{A}$$

- Coarse-grain by defining the particle density to be

$$\rho(x) = \sum_{k=1 \ldots N} \delta(x - x_k). \tag{16}$$

Then (**??**) is equivalent to $\dot{x}_j = v(x_j)$ where

$$v(x) \equiv -\omega x + \gamma \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \rho(y) \, dy, \tag{17}$$

and density is subject to conservation of mass

$$\rho_t + \nabla \cdot (\rho v) = 0. \tag{18}$$

- [Fetecau&Huang&Kolokolnikov2011]: In the limit $N \to \infty$, the steady state density of (A) is constant inside the ball of radius

$$R_0 = \sqrt{N\gamma/\omega}.$$

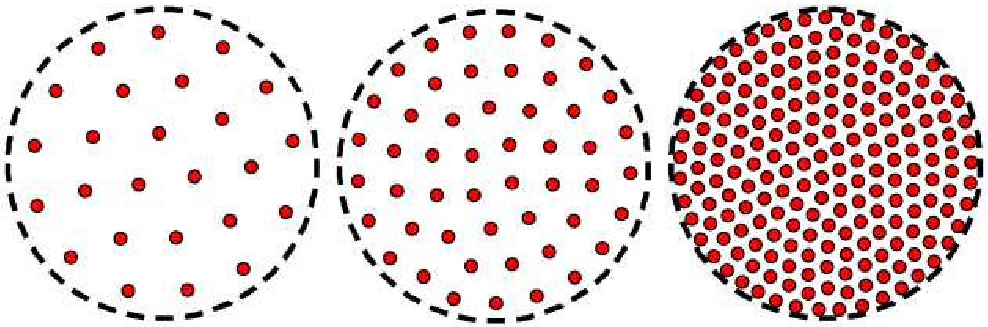
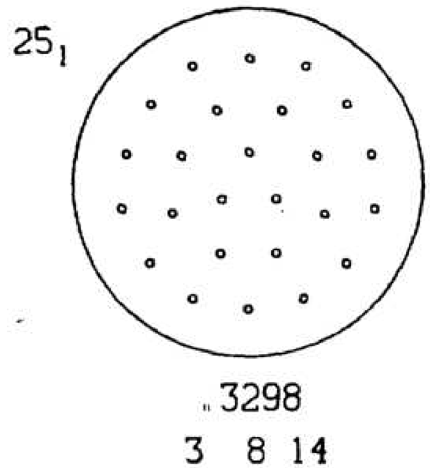


Fig. 1. Stable relative equilibria of $N = 25, 50$ and 200 vortices of equal strength. The dashed line shows the analytical prediction $R_0 = \sqrt{N\gamma/\omega}$ of the swarm radius in the $N \to \infty$ limit (see (6)).



Crystallization

Vortex model: $$\frac{dz_j}{dt} = i\sum_{k\neq j}\gamma_k \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1\dots N. \tag{V}$$

Reltive equiliria: $$z_j(t) = e^{\omega i t}\xi_j \quad \Longleftrightarrow \quad 0 = \sum_{k\neq j}\gamma_k \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} - \omega\xi_j$$

Vortex with dissipation: $$\frac{dz_j}{dt} = i\sum_{k\neq j}\gamma_k \frac{z_j - z_k}{|z_j - z_k|^2} + \mu\left(\sum_{k\neq j}\gamma_k \frac{z_j - z_k}{|z_j - z_k|^2} - \omega z_j\right) \tag{D}$$

- In many physical experiments of BEC there is damping or dissipation involved.
- ***Spectral equivalence:*** Relative equilibria ***and their stability*** are the same for (V) and (D)
- Both the vortex model and the “aggregation model” model are limiting cases of (D).
- Taking $\mu > 0$ ***stabilizes vortex dynamics!*** $chaos\ damped\ stable$
- This allows us to find stable relative equilibria numerically.

Vortex dynamics in BEC with trap

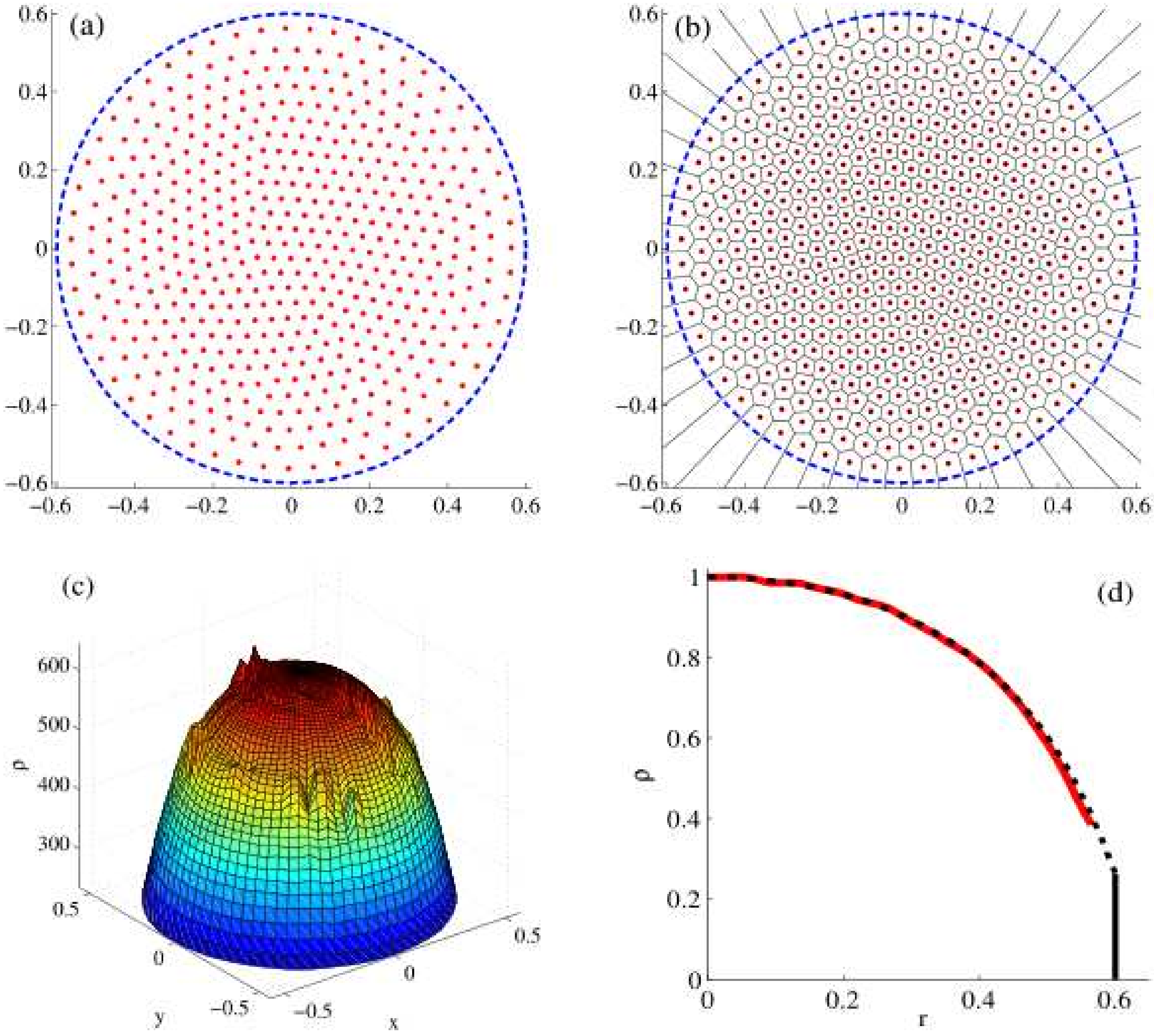
- For BEC, dynamics have extra term corresponding to prcession around the trap:

$$\dot{z}_j = \underbrace{i\frac{a}{1-r^2}z_j}_{\text{trap-interaction}} + \underbrace{ic\sum_{k\neq j}\frac{z_j - z_k}{|z_j - z_k|^2}}_{\text{self-interaction}}, \quad j = 1\dots N. \tag{19}$$

- Large N limit: ***non-uniform*** vortex lattice:

$$\rho \sim \omega - \frac{a}{(1-r^2)^2} \text{ if } r < R, \quad \rho = 0 \text{ otherwise,}$$

with $\omega = \dfrac{a}{1-R^2} + \dfrac{cN}{R^2}$



(a)
(b)
(c)
(d)
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
600
500
400
300
ρ
0.5
0
−0.5
y
x
1
0.8
0.6
0.4
0.2
r

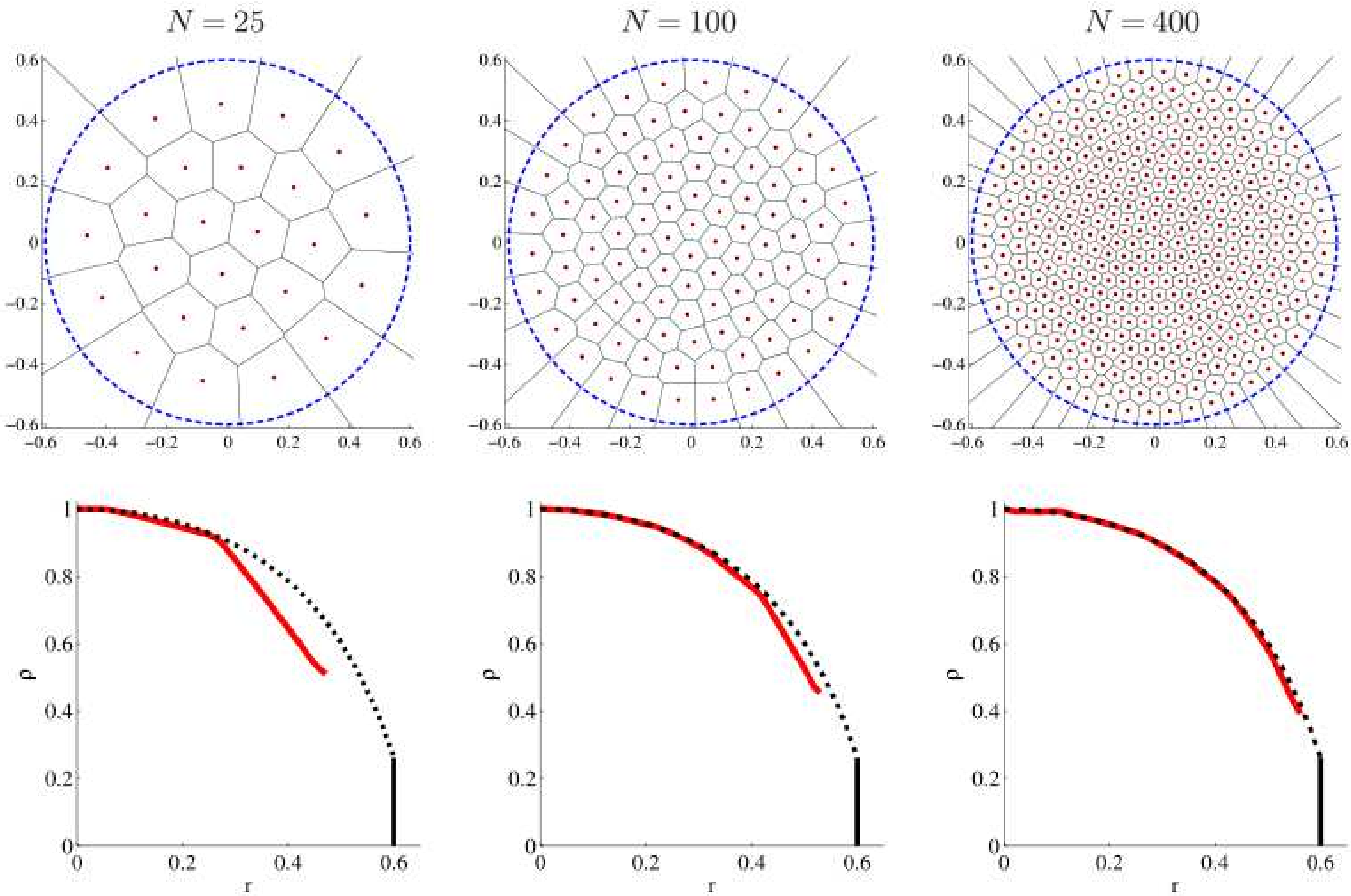
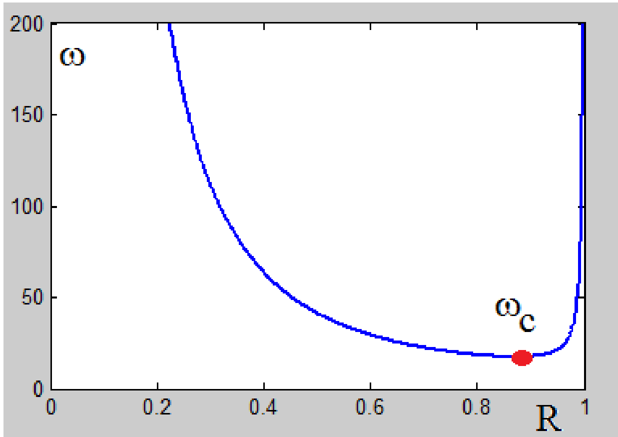


Figure 2. Top row: stable equilibrium of Eq. (2.4) with $f(r)$ as in Eq. (2.2), with N as shown in the title and with $c = 0.5/N$, $\omega = 2.95139$, $a = 1$. The dashed circle is the asymptotic boundary whose radius $R = 0.6$ is the smaller solution to Eq. (4.9). Bottom row: average of $\rho(|x|)/\rho(0)$ as a function of $r = |x|$. Solid curve corresponds to the numerical computation. Dashed curve is the formula (4.10). Vertical line is the boundary $r = R$.

Maximum N

$$\omega_c = \left(\sqrt{a} + \sqrt{cN}\right)^2; \quad R_c^2 = \frac{\sqrt{cN}}{\sqrt{a} + \sqrt{cN}}.$$



- No solutions if $\omega < \omega_c$
- Two solutions $R = R_\pm$ if $\omega > \omega_c$
  - smaller is stable, larger has negative density (unphysical).
- Corrollary: must have $N < N_{\max}$ where

$$N_{\max} = \frac{\left(\sqrt{\omega} - \sqrt{a}\right)^2}{c}. \tag{20}$$

$N+1$ problem

- N vortices of equal strength and a single vortex of a much higher strength:

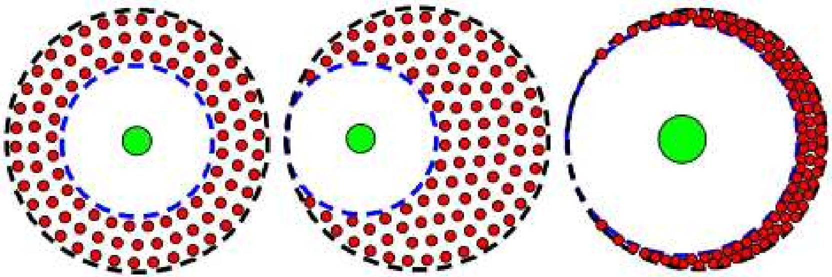
$$\frac{dx_j}{dt} = \frac{a}{N} \sum_{\substack{k=1\dots N \\ k\neq j}} \frac{x_j - x_k}{|x_j - x_k|^2} + b\frac{x_j - \eta}{|x_j - \eta|^2} - x_j, \quad j = 1 \dots N, \tag{21}$$

$$\frac{d\eta}{dt} = \frac{a}{N} \sum_{k=1\dots N} \frac{\eta - x_k}{|\eta - x_k|^2} - \eta \tag{22}$$

- Mean-field limit $N \to \infty$:

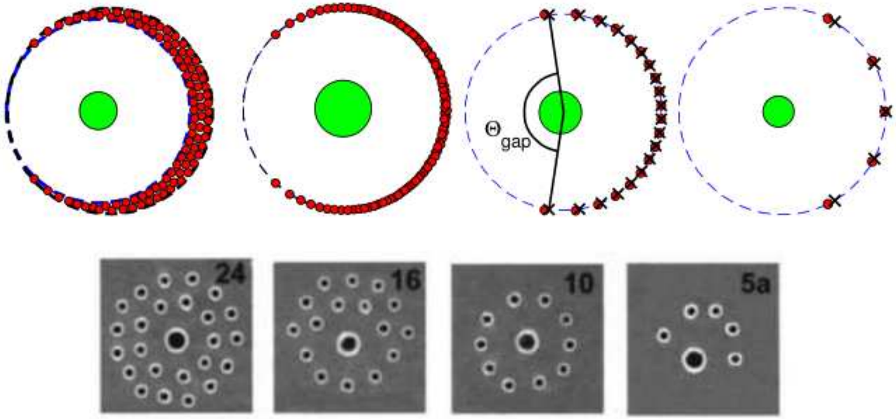
$$\begin{cases} \rho_t + \nabla \cdot (\rho \nabla v) = 0; \\ v(x) = a \int_{\mathbb{R}^2} \rho(y) \frac{x-y}{|x-y|^2} dy + b\frac{x-\eta}{|x-\eta|^2} - x \\ \frac{d\eta}{dt} = a \int_{\mathbb{R}^2} \rho(y) \frac{\eta - y}{|\eta - y|^2} dy - \eta \end{cases}. \tag{23}$$

- **Main result:**. Define $R_1 = \sqrt{b}$, $R_0 = \sqrt{a+b}$ and suppose that η is any point such that $B_{R_1}(\eta) \subset B_{R_0}(0)$. Then the equilibrium solution for (23) is constant inside $B_{R_0}(0) \backslash B_{R_1}(\eta)$ and is zero outside.



- Unlike the $N+0$ problem, the relative equilibrium for the $N+1$ problem is non-unique: any choice of η yields a steady state as long as $|\eta| < R_0 - R_1$.

Degenerate case: big central vortex



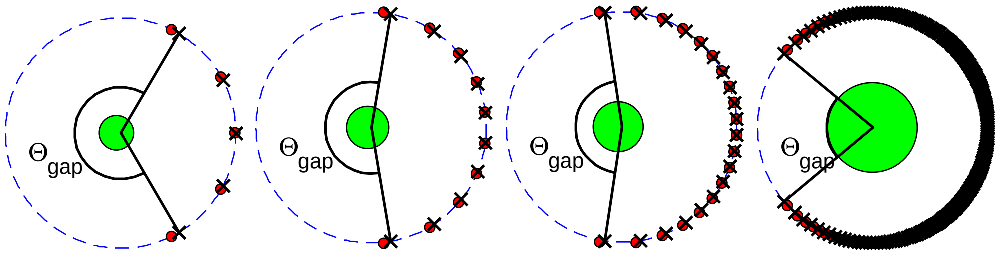
- Small vortices are constrained to a ring of radius R_0. with big vortex at the center.
- ***Non-uniform*** distribution of small particles!
- Question: Determine the size of the gap Θ_{gap}.

- **Main Result:**

$$\Theta_{gap} \sim CN^{-1/3}.$$

where the constant $C = 8.244$ satisfies

$$\left(8-6u+2u^3\right)\ln\left(u-1\right) = 3u\left(u^2-4\right); \quad C = 2\left(\frac{6\pi(2-u)}{u\left(u^2-1\right)}\right)^{1/3}$$



Sketch of proof

- [Barry+Wayne, 2012]: Set $x_j(t) \sim R_0 e^{i\theta_j(t)}$ then at leading order we get

$$\frac{d\theta_j}{dt} = \frac{1}{N} \sum_{k \neq j} \left(\frac{\sin(\theta_j - \theta_k)}{2 - 2\cos(\theta_j - \theta_k)} - \sin(\theta_j - \theta_k) \right). \tag{24}$$

- In the mean-field limit $N \to \infty$, the density distribution $\rho(\theta)$ for the angles θ_j satisfies

$$\begin{cases} \rho_t + (\rho v_\theta)_\theta = 0, \\ v(\theta) = PV \displaystyle\int_{-\pi}^{\pi} \rho(\phi) \left(\frac{\sin(\theta - \phi)}{2 - 2\cos(\theta - \phi)} - \sin(\theta - \phi) \right) d\phi, \end{cases} \tag{25}$$

  where PV denotes the principal value integral, and $\int_{-\pi}^{\pi} \rho = 1$.

- [Barry, PhD Thesis]: Up to rotations, the steady state density $\rho(\theta)$ for which $v = 0$ must be of the form

$$\rho(\theta) = \frac{1}{2\pi} (1 + \alpha \cos\theta). \tag{26}$$

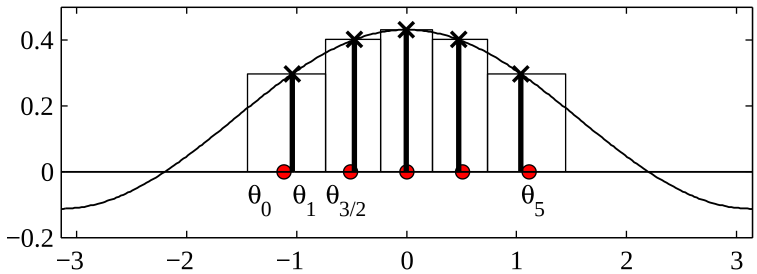
  This follows from (25) and (formal) expansion

$$\frac{\sin t}{2 - 2\cos t} - \sin t = \sin(2t) + \sin(3t) + \sin(4t) + \dots$$

- α is free parameter in the continuum limit.

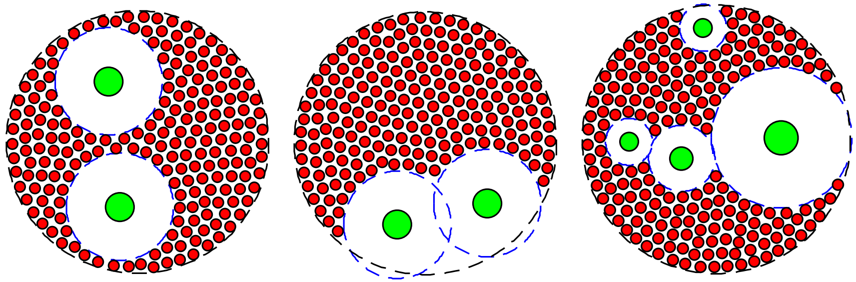
- For discrete N, particle positions satisfy

$$\int_{\theta_{j-1}}^{\theta_j} \frac{1}{2\pi} \left(1 + \alpha \cos\theta\right) d\theta = \frac{1}{N}$$



To estimate Φ_{gap}, choose θ_1 so that $v(\theta_1) \sim 0$. See our paper for hairy details.

$N + K$ problem



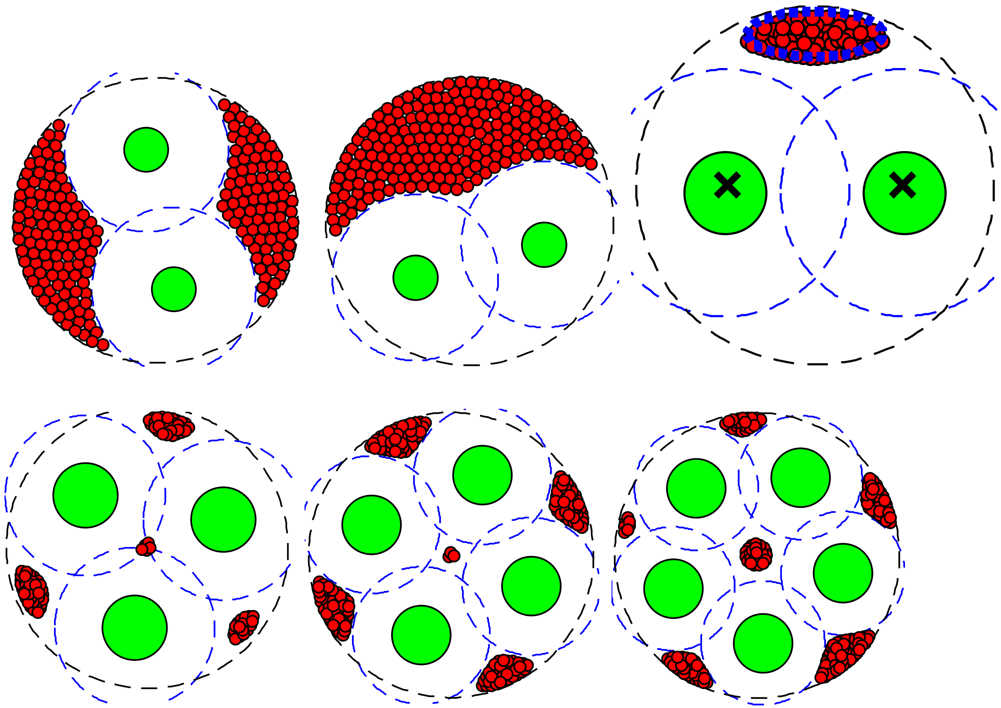
$$v(x) = a \int_{\mathbb{R}^2} \rho(y) \frac{x-y}{|x-y|^2} dy + \sum_{k=1\dots K} b_k \frac{x-\eta_k}{|x-\eta_k|^2} - x,$$

$$\frac{d\eta_j}{dt} = a \int_{\mathbb{R}^2} \rho(y) \frac{\eta_k - y}{|\eta_k - y|^2} dy + \sum_{\substack{k=1\dots K \\ k\neq j}} b_k \frac{\eta_j - \eta_k}{|\eta_j - \eta_k|^2} - \eta_j,$$

$$j = 1 \dots K.$$

Main result: Let $R_k = \sqrt{b_k}$, $k = 1 \dots K$ and $R_0 = \sqrt{a + b_1 + \dots + b_K}$. Suppose $\eta_1 \dots \eta_K$ are such $B_{R_1}(\eta_1) \dots B_{R_K}(\eta_K)$ are all disjoint and are contained inside $B_{R_0}(0)$. The equilibrium density is constant inside $B_{R_0}(0) \setminus \bigcup_{k=1}^K B_{R_k}(\eta_k)$ and is zero outside.

$N + K$ problem, with very large K vortices



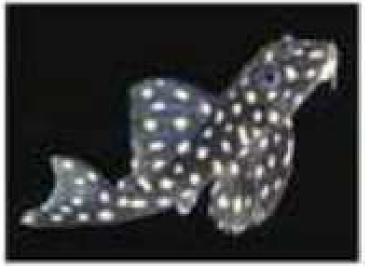
- The ***blue ellipse*** is described by the reduced system

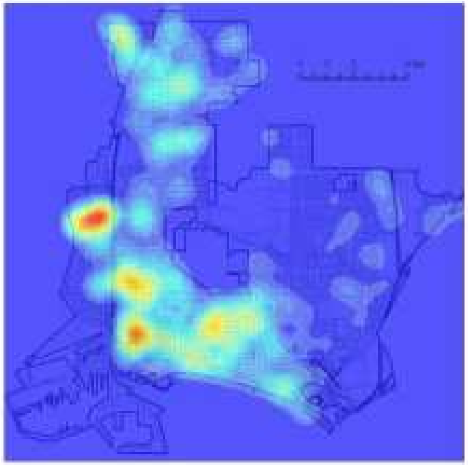
$$\frac{d\xi_j}{dt} = \frac{1}{N} \sum_{\substack{k=1\dots N \\ k \neq j}} \frac{1}{\overline{\xi_j - \xi_k}} + \frac{1}{2}\bar{\xi}_k - \xi_k \tag{27}$$

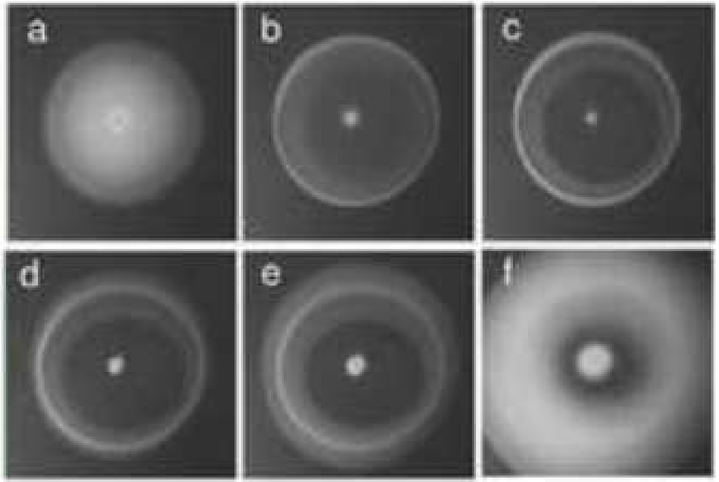
- From [K, Huang, Fetecau, 20011], its axis ratio is 3.

Spot solutions in Reaction-diffusion systems

seashells * fish * crime hotspots in LA * stressed bacterial colony





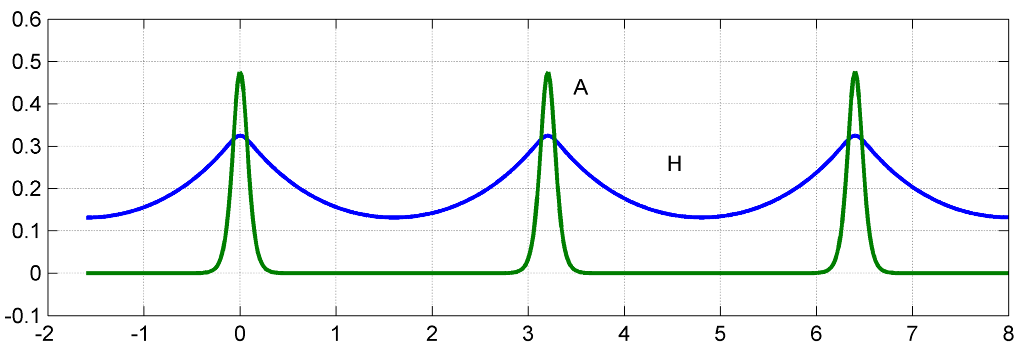


Classical Gierer-Meinhardt model

$$A_t = \varepsilon^2 \Delta A - A + \frac{A^2}{H}; \qquad \tau H_t = D\Delta H - H + A^2$$

- Introduced in 1970's to model cell differentation in hydra
- Mostly of mathematical interest: one of the simplest RD systems
- Has been intensively studied since 1990's [by mathematicians!]
- Key assumption: ***separation of scales***

$$\varepsilon \ll 1 \text{ and } \varepsilon^2 \ll D.$$



- Roughly speaking, H is constant on the scale of A so the steady state looks ”roughly” like $A(x) \sim Cw\left(\frac{x-x_0}{\varepsilon}\right)$ where

$$\Delta w - w + w^2 = 0.$$

- Questions: What about stability? What about location of the spike x_0?

“Classical” Results in 1D:

- Wei 97, 99, Iron+Wei+Ward 2000: Stability of K spikes in the GM model in one dimension

- Two types of possible instabilitities: structural instabilities or translational instabilities

- Structural instabilities (large eigenvalues) lead to spike collapse in $O(1)$ time

- Translational instabilities can lead to ”slow death”: spikes drift over large time scales

- **Main result 1**: There exists a sequence of thresholds D_K such that K spikes are stable iff $D < D_K$.

- **Main result 2:** Slow dynamics of K spikes is described by an ODE with $2K$ variables (spike heights and centers) subject to K algebraic constraints between these variables.

Large eigenvalues

- Careful derivation leads to a ***nonlocal eigenvalue problem*** (NLEP) of the form

$$\lambda\phi = \Delta\phi + (-1+2w)\,\phi - \chi w^2 \frac{\int w\phi}{\int w^2}; \quad \chi := \frac{4\sinh^2\left(\frac{1}{\sqrt{D}}\right)}{2\sinh^2\left(\frac{1}{\sqrt{D}}\right) + 1 - \cos\left[\pi(1-1/K)\right]}$$

- **Key theorem (Wei, 99):** $\mathrm{Re}(\lambda) < 0$ iff $\chi < 1$

- **Corrollary**: On a domain $[-1,1]$, large eigenvalues are stable iff $D < D_{K,\text{large}}$ where

$$D_{K,\text{large}} = \frac{1}{\operatorname{arcsinh}^2(\sin 2\pi/K)}$$

- When unstable, this can lead to ***competition instability.***

- Movies: stable; unstable

Small eigenvalues

- Causes a ***very slow drift***
- Iron-Ward-Wei 2000: The slow dynamics of the system can be reduced to a coupled algbraic-differential system of ODEs
- Movie: slow drift

Two dimensions

- Structural stability is similar

- Dynamics [Ward et.al, 2000, K-Ward, 2004, K-Ward 2005]:

$$\frac{dx_0}{dt} \sim -\frac{4\pi\varepsilon^2}{\ln \varepsilon^{-1} + 2\pi R_0} \nabla R_0$$

  where

$$R_0 = \lim_{x \to x_0} \left[G(x, x_0) + \frac{1}{2\pi} \ln(|x - x_0|) \right];$$

$$\nabla R_0 = \lim_{x \to x_0} \nabla_x \left[G(x, x_0) + \frac{1}{2\pi} \ln(|x - x_0|) \right];$$

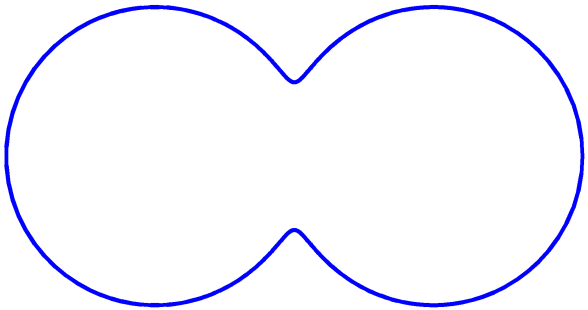
$$\Delta G - \frac{1}{D} G = -\delta\left(x - x_0\right) \text{ on } \Omega; \quad \partial_n G = 0 \text{ on } \partial\Omega$$

- Equilibrium location x_0 satisfies $\nabla R_0 = 0$, occurs at the extremum of the regular part of the Neumann's Green's function

Dumbbell-shaped domain

- QUESTION: Suppose that a domain has a dumb-bell shape. Where will the spike drift??

- What are the possible equilibrium locations for a single spike?



Small D limit

- If D is very small, $R_0(x_0) \sim C(x_0) \exp\left(-\frac{1}{\sqrt{D}} |x_0 - x_m|\right)$ where x_m is the point on the boundary closest to x_0

- This means that R_0 is ***minimized at the point furthest away from the boundary when*** $D \ll 1$
  - In the limit $\varepsilon^2 \ll D \ll 1$, the spike drifts towards the point furthest away from the boundary.
  - For a dumbell-shaped domain above, the three possible equilibria are at the ”centers” of the dumbbells (stable) and at the center of the neck (unstable saddle point)
  - For multiple spikes, their locations solve ”ball-packing problem”.

- Movie: $D = 0.03, \varepsilon = 0.04$

Large D limit

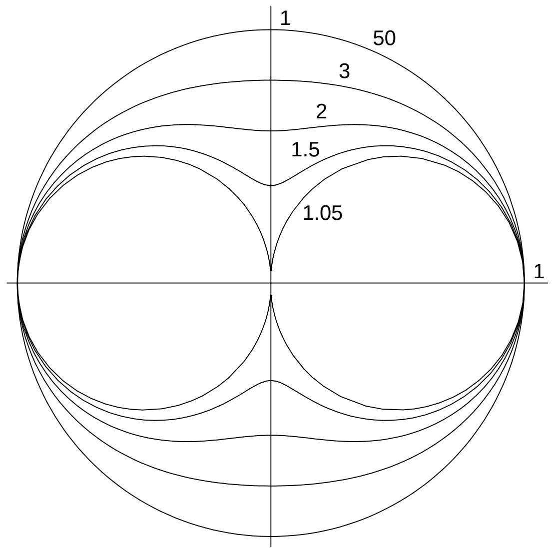
- We get the ***modified Green's function:***

$$\Delta G_m - \frac{1}{|\Omega|} = -\delta(x - x_0) \text{ inside } \Omega, \quad \partial_n G = 0 \text{ on } \partial\Omega;$$

$$R_{m0} = \lim_{x \to x_0} \left[G_m(x, x_0) + \frac{1}{2\pi} \ln(|x - x_0|) \right].$$

- [K, Ward, 2003]: For a domain which is an analytic mapping of a unit disk, $\Omega = f(B)$, we derive an ***exact formula*** for ∇R_{m0} in terms of the residues of $f(z)$ *outside* the unit disk.

- Take $f(z) = \dfrac{(1 - a^2)z}{z^2 + a^2}; \qquad x_0 = f(z_0):$



Then

$$\nabla R_{m0}(x_0) = \frac{\nabla s(z_0)}{f'(z_0)}$$

where

$$\nabla s(z_0) = \frac{1}{2\pi}\left(\begin{array}{c} \frac{z_0}{1-|z_0|^2} - \frac{\left(\bar{z}_0^2+3a^2\right)\bar{z}_0}{\bar{z}_0^4-a^4} + \frac{a^2\bar{z}_0}{\bar{z}_0^2a^2-1} + \frac{\bar{z}_0}{\bar{z}_0^2-a^2} \\ -\frac{(a^4-1)^2(|z_0|^2-1)(z_0+a^2\bar{z}_0)(\bar{z}_0^2+a^2)}{(a^4+1)(\bar{z}_0^2a^2-1)(z_0^2-a^2)(\bar{z}_0^2-a^2)^2} \end{array} \right)$$

- Corrollary: for above Ω, ∇R_{m0} has a unique root at the origin!
  - In the limit $D \gg 1$, all spikes will drift towards the neck.

- Complex bifurcation diagram as D is increased.

- Movie: $\varepsilon = 0.05$, $D = 0.1$; $D = 1$.

"Huge" D

- In the limit $D \to \infty$, (Shadow limit), an interior spike is unstable and moves towards the boundary [Iron Ward 2000; Ni, Polácik, Yanagida, 2001].

- For ***exponentially large but finite*** $D = O(\exp(-C/\varepsilon))$, boundary effects will compete with the Green's function.

- [K, Ward, 2004]: Define

$$\sigma := \frac{\varepsilon}{2} \ln \left(\frac{C_0}{|\Omega|} D \varepsilon^{-1/2} \right); \quad C_0 \approx 334.80;$$

  Then the spike will move towards the boundary whenever its distance from the closest point of the boundary is at most σ; otherwise it will move away from the boundary.

- Movies: $\varepsilon = 0.05$, $D = 10$; $D = 100$

Spike dynamics inside a disk

In the limit $\varepsilon \ll 1, D \gg 1$, inside the disk we get

$$C\frac{dx_j}{dt} \sim \underbrace{2\sum_{k\neq j}\frac{x_j - x_k}{|x_j - x_k|^2} - \sum_k x_j}_{inter-particle\ force} + \underbrace{\sum_k \frac{x_j - x_k/|x_k|^2}{\left|x_j - x_k/|x_k|^2\right|^2} - \sum_k \frac{-x_j|x_k|^2 + x_k|x_j|^2}{\left|x_j|x_k|^2 - x_k\right|^2}}_{reflection\ in\ the\ boundary\ of\ unit\ disk}.$$

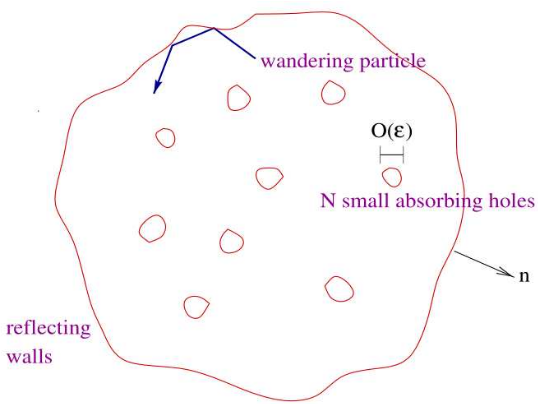
- The first two terms are identical to vortex stability model!
- The last two terms represent “reflection in the wall”
- Just like for vortex model, ***the steady state consists of uniformly-distributed particles inside the domain!***
- Movies: disk; dumbbell.

Mean first passage time (ice fishing)

- Question: Suppose you want to catch a fish in a lake covered by ice. Where do you drill a hole to maximize your chances?

- Related questions: cell signalling; oxygen transport in muscle tissues; cooling rods in a nuclear reactor...

- Consider N non-overlapping small "holes" each of small radius ε. A particle is performing a random walk inside the domain Ω. If it hits a hole, it gets destroyed; if it hits a boundary, it gets reflected. Question: what is the expected lifetime of the wondering particle? How do we place the holes to minimize this lifetime [i.e. catch the fish, cool the nuclear reactor...]?



- The expected lifetime is proportional to $1/\lambda$ where λ is the smallest eigenvalue of the problem:

$$\Delta u + \lambda u = 0 \text{ inside } \Omega \backslash \Omega_p; \quad u = 0 \text{ on } \partial\Omega_p;\ \partial_n u = 0 \text{ on } \partial\Omega$$

where $\Omega_p = \bigcup_{i=1}^N \Omega_\varepsilon$.

- [K-Ward-Titcombe, 2005]: The smallest eigenvalue is given by

$$\lambda \sim \frac{2\pi N}{\ln\frac{1}{\varepsilon}} \left(1 - \frac{2\pi}{\ln\frac{1}{\varepsilon}} p(x_1, \ldots x_N) + O\left(\frac{1}{\left(\ln\frac{1}{\varepsilon}\right)^2}\right)\right)$$

where

$$p(x_1, \ldots x_N) := \sum\sum G_{ij};$$

$$G_{ij} = \begin{cases} G_m(x_i, x_j) & \text{if } i \neq j \\ R_m(x_i, x_i) & \text{if } i = j \end{cases}$$

$$\Delta G_m(x, x') - \frac{1}{|\Omega|} = -\delta(x - x') \text{ inside } \Omega, \quad \partial_n G = 0 \text{ on } \partial\Omega; \quad R_m \equiv reg.part$$

- For a unit disk:

$$2\pi G_m(x, x') = -\ln|x - x'| - \ln\left|x\,|x'| - \frac{x'}{|x'|}\right| + \frac{1}{2}\left(|x|^2 + |x'|^2\right)$$

$$2\pi R_m(x, x') = -\ln\left|x\,|x'| - \frac{x'}{|x'|}\right| + \frac{1}{2}\left(|x|^2 + |x'|^2\right)$$

- The optimum trap placement is at the minimum of $p(x_1, \ldots x_N)$

Disk domain, N holes

We need to minimize

$$p(x_1 \ldots x_N) = -\sum_{j \neq k} \ln |x_j - x_k| - \sum_{j,k} \left(\ln \left| x_j - \frac{x_k}{|x_k|^2} \right| + \ln |x_k| \right) + \frac{1}{2} \sum_{j,k} \left(|x_j|^2 + |x_k|^2 \right)$$

Gradient flow is uniform swarm model plus two extra terms

$$\frac{dx_j}{dt} = 2 \sum_{k \neq j} \frac{x_j - x_k}{|x_j - x_k|^2} - \sum_k x_j + \sum_k \frac{x_j - x_k / |x_k|^2}{\left| x_j - x_k / |x_k|^2 \right|^2} - \sum_k \frac{-x_j |x_k|^2 + x_k |x_j|^2}{\left| x_j |x_k|^2 - x_k \right|^2}.$$

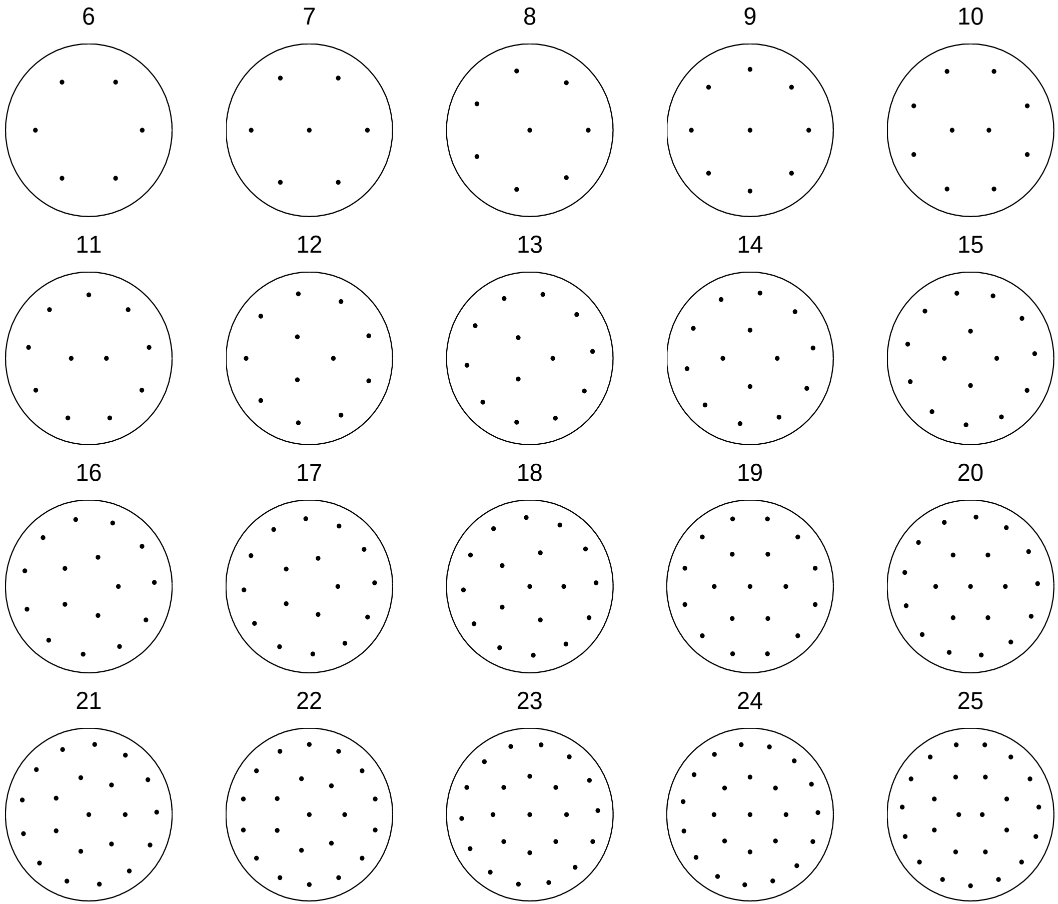
Particles on a ring: $x_k = re^{ik2\pi/N}$. The min occurs when

$$\frac{r^{2N}}{1 - r^{2N}} = \frac{N-1}{2N} - r^2$$

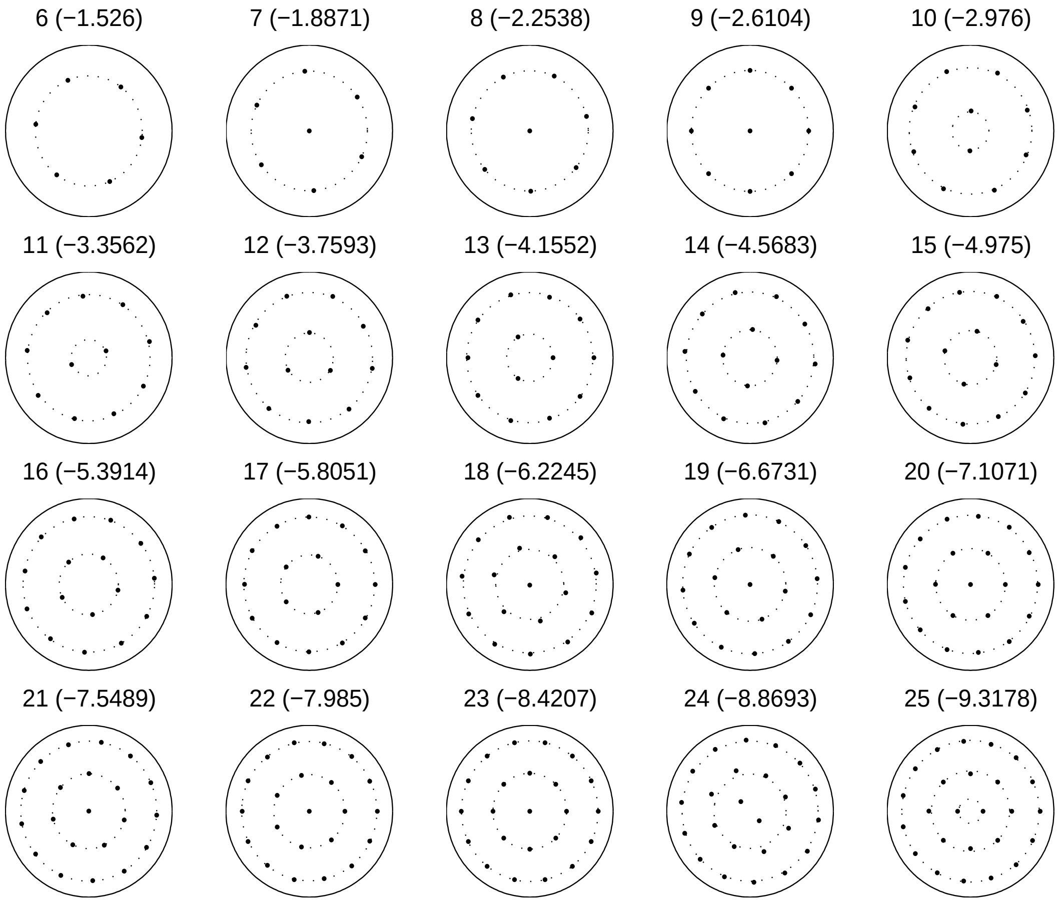
Note that $r \to 1/\sqrt{2}$ as $N \to \infty$; the optimal ring divides the unit disk into two equal areas.

Particles on 2,3,...m rings: Similar results are derived with complicated but numerically useful formulas.

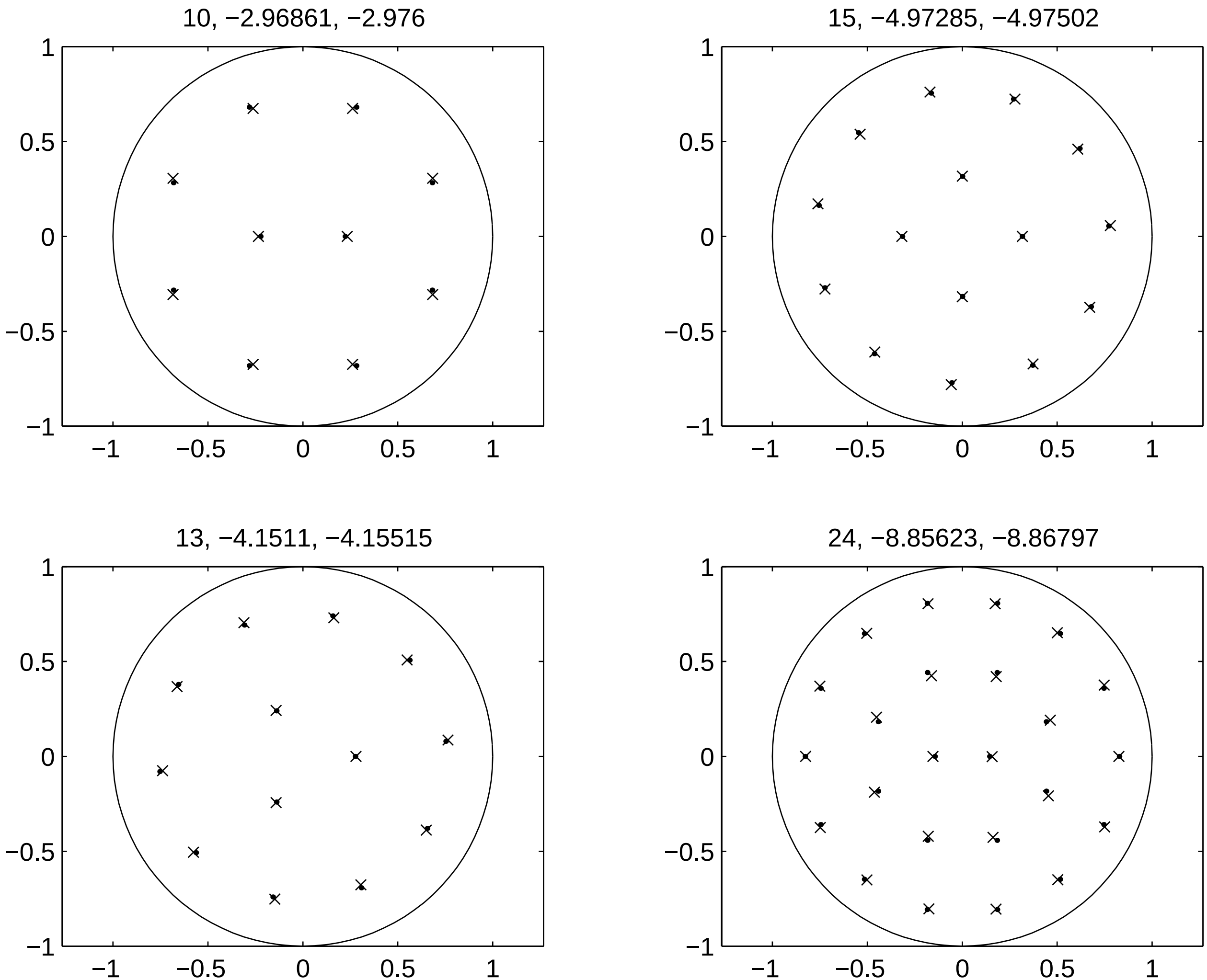
Constrained optimization on up to 3 rings



Full optimization of K traps



Comparison



10, −2.96861, −2.976
15, −4.97285, −4.97502
13, −4.1511, −4.15515
24, −8.85623, −8.86797
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1

Conclusion

- We looked at three very different problems: vortex dynamics; spike dynamics and first mean-passage time

- All three problems reduce to ***nonlocal particle aggregation model*** with Newtonial repulsion

- In the limit of large number of particles, the steady state approaches a ***uniform distribution.***

- Spectral equivalence of aggregation and vortex model shows stability

These papers are available for download from my website: `http://www.mathstat.dal.ca/~tkolokol`

Thank you! Any questions?