Curved interfaces in perturbed Allen-Cahn system

Theodore Kolokolnikov

Joint work with

David Iron
John Rumsey
Juncheng Wei
Introduction

We consider the perturbed Allen-Cahn equation:

\[
\begin{align*}
    u_t &= \varepsilon^2 \Delta u + f(u) + \varepsilon g(u), \quad x \in \Omega \subset \mathbb{R}^2, \\
    \partial_n u &= 0, \quad x \in \partial \Omega.
\end{align*}
\] (PAC)

Here, \( \Omega \) is a smooth two-dimensional domain and \( f(u) \) is a smooth function having the following properties:

- \( f \) has three roots \( u_- < u_0 < u_+ \) with \( f'(u_\pm) < 0 \)
- \( \int_{u_-}^{u_+} f(u) \, du = 0 \)

and \( g(u) \) is any smooth function with \( \int_{u_-}^{u_+} g(u) \, du \neq 0 \).
Some known results

Standard form:
\[
\begin{cases}
  u_t = \varepsilon^2 \Delta u - 2(u - \varepsilon a)(u^2 - 1), & x \in \Omega \subset \mathbb{R}^2 \\
  \partial_n u = 0, & x \in \partial \Omega
\end{cases}
\]

Standard A-C corresponds to \( a = 0 \):

- In 1-D, the steady state is given by \( u = \pm \tanh(x/\varepsilon) \).
- In 2-D, the profile is 1-dimensional in some direction; the zero set \( u = 0 \) is a straight line, intersects boundary transversally.
- Such straight interface is stable (unstable) provided it is a local min (max) of the distance function. [Kowalczyk, 05]
- Time dependent solution evolves by mean curvature law until the interface merges with the boundary or becomes straight. [RSK, 89]
Effect of perturbation: numerics
Let $U_0(z)$ be a solution to

$$U''_0(z) + f(U_0) = 0, \quad U \to u_{\pm} \text{ as } z \to \pm\infty.$$ 

and define

$$\hat{R} = -\frac{\int_{-\infty}^{\infty} U''_0(z) \, dz}{\int_{u_-}^{u_+} g(u) \, du}$$  \hspace{1cm} (1)$$

Suppose that there exists a circle of radius $\hat{R}$ which intersects $\partial \Omega$ orthogonally, and let $p$ be its center. Then in the limit $\varepsilon \to 0$ we have

$$u(x) \sim U_0 \left( \frac{\hat{R} - |p - x|}{\varepsilon} \right), \quad \varepsilon \to 0$$  \hspace{1cm} (2)$$

any solution to (PAC) of the form (2) must satisfy (1).
Derivation for cone-shaped domain

- Solution is radially symmetric:
  \[ \varepsilon^2 u_{rr} + \frac{1}{r} u_r + f(u) + \varepsilon g(u) = 0 \]

- Solvability condition determines radius
Main stability result

Consider an interface at an equilibrium whose radius is $\hat{R}$. Let $\ell$ be its length let $\kappa_+, \kappa_-$ be the curvatures of the boundary at the points which intersect the interface. Consider the stability problem associated with (PAC),

\begin{equation}
\begin{cases}
\lambda \phi = \varepsilon^2 \Delta \phi + f'(u) \phi + \varepsilon g'(u) \phi, & x \in \Omega \\
\partial_n \phi = 0, & x \in \partial \Omega.
\end{cases}
\tag{EP}
\end{equation}

In the limit $\varepsilon \to 0$, we have $\lambda = \varepsilon^2 \lambda_0$ where $\lambda_0$ satisfies

$$\lambda_0 = \frac{1}{\hat{R}^2} - \mu^2 \quad \text{where} \quad \tan (\ell \mu) = -\frac{\mu (\kappa_+ + \kappa_-)}{\mu^2 - \kappa_+ \kappa_-} \tag{3}$$

or

$$\arctan \left( \frac{-\kappa_+}{\mu} \right) + \arctan \left( \frac{-\kappa_+}{\mu} \right) = \ell \mu. \tag{4}$$
Example

\[ u_t = \varepsilon^2 \Delta u - 2(u - \varepsilon a)(u - 1)(u + 1); \quad a = 0.55, \quad \varepsilon = 0.06; \]
\[ \kappa_- = -1.25, \quad \kappa_+ = -0.667, \quad l = 0.6486. \]

\[ \hat{R}_{\text{theory}} = 1/(2a) = 0.9091; \quad \hat{R}_{\text{numerical}} = 0.9066 \]

\[ \lambda_{\text{theory}} = 0.00506. \quad \lambda_{\text{numerical}} = 0.00504. \]
Instability on a cone domain

Radially symmetric case:

- Expand

\[ \phi = \Phi_0(z) + \varepsilon \Phi_0 + \cdots \]

\[ r = \hat{R} + \varepsilon z \]

- Apply solvability condition

- End result:

\[ \lambda_0 = \frac{1}{\hat{R}^2} > 0. \]

- Interface is unstable on a cone [or any convex domain]
Geometric eigenvalue problem

... Equivalently, \( \lambda = \varepsilon^2 \lambda_0 \) where \( \lambda_0 \) satisfies

\[
\begin{cases}
    w'' + (\lambda_0 - \hat{R}^{-2})w = 0 \\
    w'(-\ell/2) + \kappa_- w(-\ell/2) = 0 \\
    w'(-\ell/2) + \kappa_+ w(\ell/2) = 0.
\end{cases}
\]

(GEP)

The standard AC model corresponds straight interface, \( \hat{R}^{-1} = 0 \). In this case (GEP) is the same as the formula derived by Kowalczyk (2005).

In the case \( \hat{R}^{-1} = 0 \), stability threshold \( \lambda_0 = 0 \) occurs when \( l + \kappa_-^{-1} + \kappa_+^{-1} = 0 \). Geometrically, the circles tangent to the boundaries are concentric.

QUESTION: What is the threshold in general case \( (\hat{R}^{-1} \neq 0) \)?
Geometric criterion for stability

Stability $\iff R'(s) < 0$ whenever $R = \hat{R}$

Example: If $\hat{R} = 1$ then curve $c$ represents the location of a stable interface, whereas curves $a$ and $e$ correspond to unstable interfaces.
Derivation of Geometric Criterion

\[ R = \frac{R_+ (1 - \cos \theta_+) + h_+}{\sin \theta_+}. \]

\[ R' = 0 \iff \arctan \left( \frac{R}{R_{\pm}} \right) = \theta_{\pm} \]

\[ \theta_+ = \ell_+/R, \quad \theta_- = \ell_-/R \quad \text{and} \]
\[ \ell = \ell_+ + \ell_- \]

\[ R' = \iff \lambda_0 = 0 \]

For a cone domain, \( \lambda_0 = 1/\hat{R}^2 > 0 \).

By continuity, \( \lambda_0 > 0 \) whenever \( R' > 0 \).
Open question: Stability of tractrix

In general, we have

\[ R \frac{d\theta}{ds} = p_1' \sin \theta - p_2' \cos \theta. \]

where \( p, \theta, R \) are functions of \( s \), and the boundary of the domain is given by \( p + R(\cos \theta, \sin \theta) \).

If \( R \) is constant then every circle that intersects the boundary orthogonally has the same radius. The resulting curve is a tractrix given by

\[ x = \hat{R}(-t + \tanh(t)), \quad y = \hat{R} \text{sech}(t). \]

\( \lambda_0 = 0, \lambda = O(\varepsilon^3)? \) Open question: Determine stability.
Open question: Corner junctions

\[ R_1 = \frac{h}{\sin \theta}, \quad R_2 = R_1 + d, \quad R_3 = h + d \sin \theta \]

If we “smooth out” the corners and start with an interface at the left...

- If \( \hat{R} > R_1 \) then interface stops at left corner
- If \( R_1 > \hat{R} > R_3 \) then interface stops at right corner
- If \( \hat{R} < \min(R_1, R_3) \) then interface propagates and dies.
Open question: Corner junctions

Example: \( h = 0.2, d = 0.5, \theta = \pi/6; \) then

\[
R_1 = 0.6, R_2 = 0.9, R_3 = 0.45
\]

- Take \( \hat{R} = 0.65; \) interface stops at first corner
- Take \( \hat{R} = 0.5; \) interface stops at second corner
- Take \( \hat{R} = 0.4; \) interface goes through
- Open question: compute \( \lambda_0 \) for corner interface...