Notes for Kadison and Ringrose, Volume 1

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Note on the notes

These notes are a work in progress (begun late August 2019), and will be added to periodically when I have time to type. Intended to clarify any areas in the text I found tricky or terse. Math books are sufficiently difficult that glosses, annotations, or "helps" can be useful, particularly to oneself at a later time.

Section 2.6

In Example 2.6.11, it is stated that $\mathscr{K}_0$ is everywhere dense in $\mathscr{K}$ since it contains the characteristic function of every measurable rectangle of finite measure. Recall that $\mathscr{K} = L^2(S \times S', \mathscr{S} \times \mathscr{S}', m \times m')$. This seems pretty intuitive, but how to verify it rigorously? Here are two approaches:

(i) One approach is, instead of using measurable rectangles of finite measure, use Proposition 7.21 in Folland's real analysis to show—in the notation of that proposition—that the span of functions of the form $f \otimes g$, where $f \in C_c(X)$, $g \in C_c(Y)$ is dense in $C_c(X \times Y)$, in the uniform norm. Then this implies that L^p density on the compact set $\overline{U} \times \overline{V}$, where U, V are defined in that proposition. Then use the density of $C_c(X \times Y)$ in $L^p(X \times Y)$. This argument would circumvent the need for working with measurable rectangles, by showing directly that $\mathscr{K}_0$ is dense in $\mathscr{K}$.

(ii) Another argument is the following. We know that the simple functions $\phi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j}$ are dense in $L^2(S \times S')$, where $(m \times m')(E_j) < \infty$ for all j. Moreover, we know that the product σ-algebra $\mathscr{S} \otimes \mathscr{S}'$ is generated by the rectangles in $S \times S'$. Consider the algebra $\mathscr{A}$ of finite unions of measurable rectangles. Since $m \times m'$ is a premeasure on $\mathscr{A}$, it follows from Proposition 1.14 in Folland's Real Analysis that the measure $m \times m'$ on $\mathscr{S} \times \mathscr{S}'$ is equal to its outer measure restricted to $\mathscr{S} \times \mathscr{S}'$. Thus, for $E \in \mathscr{S} \times \mathscr{S}'$,

$$m \times m'(E) = \inf \Big\{ \sum_{1}^{\infty} m \times m'(A_j) : A_j \in \mathscr{A}, E \subset \bigcup_{1}^{\infty} A_j \Big\}.$$

Thus we can approximate the measure of E by a sum of measures of finite unions of rectangles. And of course, any finite union of rectangles can be expressed as a finite union of disjoint rectangles. Therefore, in our simple function $\sum_{j=1}^{n} a_j \mathbb{1}_{E_j}$, each $\mathbb{1}_{E_j}$ can be approximated in $L^2(S \times S')$ by a finite sum $\sum_{j=1}^{N} \mathbb{1}_{F_j}$, where each $F_j = X_j \times Y_j$ is a rectangle of finite measure; in other words, $\mathbb{1}_{E_j}$ is approximated by $\sum_{j=1}^{N} p_{\mathbb{1}_{X_j}, \mathbb{1}_{Y_j}}$, which is the density argument we desired.

Section 3.4

Theorem 3.4.16 While this theorem is interesting, it might be helpful to have a concrete example in mind to show that $C(X)$, when X is connected, for example, is not boundedly complete. Consider the family of continuous, piecewise-linear functions illustrated below. They are all defined and bounded above on the closed, connected interval $X = [0, 1]$, but they have no least upper bound in $C([0, 1])$. In fact, they converge pointwise to the function $1 - \mathbb{1}_{[.25,.75]}$, which is discontinuous at the points $x = \frac{1}{4}, x = \frac{3}{4}$.

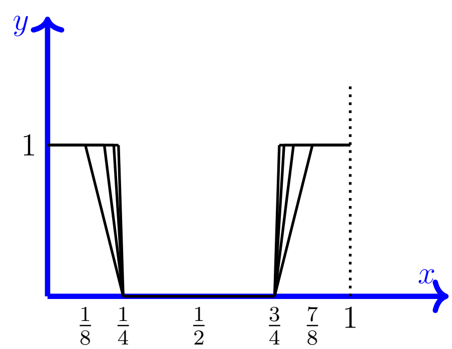

Figure 1: A family of function in $C([0, 1])$ which is bounded above, but has no least upper bound.

Recall that it may be shown that the real numbers are boundedly complete by using the fact that in $\mathbb{R}$ every Cauchy sequence converges. This argument involves constructing two Cauchy sequences that converge to the same limit, one being a sequence of upper bounds, the other a sequence of real numbers which are not upper bounds. This argument breaks down in the case of a general Banach space, where the completeness property of every Cauchy sequence converging is insufficient, as the above example illustrates.

It can, however, be shown that the self-adjoint operators in a von Neumann algebra are boundedly complete, a corollary of Lemma 5.1.4 which will be mentioned later, and which will be instrumental for the theory of von Neumann algebras.

Section 5.2

p. 315, Proof of 5.2.6 Regarding the existence of $E(\mathscr{O})$: first note that φ is order-preserving, by 4.1.8(i). Let $\varphi(f_1) \in \varphi(\mathscr{F}(\mathscr{O}))$; if $\varphi(f_1)$ is not an upper bound for $\mathscr{F}(\mathscr{O})$, then by definition of $\mathscr{F}(\mathscr{O})$ there must be some $f_2 \in \mathscr{F}(\mathscr{O})$ such that $\varphi(f_1) \leq \varphi(f_2)$; moreover, $\varphi(f_1)$ and $\varphi(f_2)$ are self-adjoint, because f_1 and f_2 are real-valued. If $\varphi(f_2)$ is not an upper bound for $\varphi(\mathscr{F}(\mathscr{O}))$, then there exists an f_3 with $\varphi(f_2) \leq \varphi(f_3)$, and so on. We obtain a monotone-increasing sequence $\{\varphi(f_n)\}$ in $\varphi(\mathscr{F}(\mathscr{O}))$ such that $0 \leq \varphi(f_n) \leq I$ for all n. By Lemma 5.1.4, $\varphi(f_n) \overset{\text{SOT}}{\to} H$ for some self-adjoint $H \in \mathscr{A}$, and moreover, $H = \text{l.u.b.}\{\varphi(f_n)\}$. Then $E(\mathscr{O})$ is defined to be this H.

Page 315 "regular Borel measure": we note that K& R use this term for what is now commonly called a *Radon measure*.

From p. 316: the range projection of $\varphi(f_0)$ is a subprojection of $E(\mathscr{O})$: This follows from Lemma 5.1.5: since $f_0 = \|f\|^{-1}|f|$ we have $0 \leq f_0 \leq 1$, with f_0 vanishing on $X \setminus \mathscr{O}$; hence $f_0 \in \mathscr{F}(\mathscr{O})$, and $0 \leq \varphi(f_0) \leq I$. And clearly $f_0^{1/n} \in \mathscr{F}(\mathscr{O})$

as well, where $E(\mathscr{O})$ is the l.u.b. of $\varphi\big(\mathscr{F}(\mathscr{O})\big)$ in $\mathscr{A}$. Hence

$$\begin{aligned} R\big(\varphi(f_0)\big) &= \text{SOT-}\lim \varphi(f_0)^{1/n} \\ &= \text{SOT-}\lim \varphi\big((f_0)^{1/n}\big) \\ &\leqslant E(\mathscr{O}). \end{aligned}$$

From p. 316: $\varphi(f_0)$ and $\varphi(f)$ have the same range projection. Kadison and Ringrose claim that $R\big(\varphi(f_0)\big) = R\big(\varphi(f)\big)$, where R denotes the range projection. Since $f_0 = \|f\|^{-1}|f|$, and $\varphi(f_0) = \|f\|^{-1}\varphi(|f|)$, this is equivalent to showing that $\varphi(|f|) = \varphi(f_+) + \varphi(f_-)$ and $\varphi(f) = \varphi(f_+) - \varphi(f_-)$ have the same range projection, where f_+ and f_- denote the positive and negative parts of f, respectively. This is to say, we must show that $\operatorname{range}\big(\varphi(f_+) + \varphi(f_-)\big) = \operatorname{range}\big(\varphi(f_+) - \varphi(f_-)\big)$. *I could not see how to do this.* I propose the following alternative argument: In order to proceed with the proof we must simply show that $R\big(\varphi(f)\big) \leqslant E(\mathscr{O})$ for f as described earlier in the proof ($f \in C(X)$ real-valued and vanishing on $X/\mathscr{O}$). Note that $0 \leq \|f\|^{-1}f_+ \leq f_0 \leq 1$, whence $0 \leq \varphi\big(\|f\|^{-1}f_+\big) \leq I$, whence by Lemma 5.1.5,

$$\begin{aligned} R\big(\varphi(f_+)\big) &= R\big(\varphi(\|f\|^{-1}f_+)\big) \\ &= \text{SOT-}\lim \varphi\big(\|f\|^{-1}f_+\big)^{1/n} \\ &= \text{SOT-}\lim \varphi\big((\|f\|^{-1}f_+)^{1/n}\big) \\ &\leqslant E(\mathscr{O}). \end{aligned}$$

Note that this line of argument is also used to show $R\big(\varphi(f_0)\big) \leqslant E(\mathscr{O})$. The same argument shows that $R\big(\varphi(f_-)\big) \leqslant E(\mathscr{O})$; hence $R\big(\varphi(f_+)\big) \vee R\big(\varphi(f_-)\big) \leqslant E(\mathscr{O})$.

We clearly have, for any $x \in \mathscr{H}$, $\big(\varphi(f_+) - \varphi(f_-)\big)(x) = \varphi(f_+)(x) + \varphi(f_-)(-x)$. Thus we have

$$\begin{aligned} \operatorname{range}\big(\varphi(f_+) - \varphi(f_-)\big) &\subseteq \operatorname{range}\big(\varphi(f_+)\big) + \operatorname{range}\big(\varphi(f_-)\big) \\ &\subseteq \operatorname{range}\big(\varphi(f_+)\big) \vee \operatorname{range}\big(\varphi(f_-)\big) \\ &= \operatorname{range}\big(R(\varphi(f_+))\big) \vee \operatorname{range}\big(R(\varphi(f_-))\big) \\ &= \operatorname{range}\big(R(\varphi(f_+)) \vee R(\varphi(f_-))\big). \end{aligned}$$

Thus we have

$$R\big(\varphi(f)\big) = R\big(\varphi(f_+) - \varphi(f_-)\big) \leqslant R(\varphi(f_+)) \vee R(\varphi(f_-)) \leqslant E(\mathscr{O}).$$

P. 318: "(at first, for each f in $C(X)$ vanishing at ∞, but then for each f continuous on $\mathrm{sp}(A)$ since each such agrees on $\mathrm{sp}(A)$ with some function vanishing at ∞)."

For this, simply let $\mathscr{O} \supset \mathrm{sp}(A)$, $\mathscr{O}$ open (and $\mathrm{sp}(A)$ is, of course, compact), and use Urysohn's lemma to get φ such that $\varphi = 1$ on $\mathrm{sp}(A)$ and $\varphi = 0$ on $X - \mathscr{O}$. Then $f\varphi \in C_0$, since C_0 is an ideal (two-sided), and $f\varphi = f$ on $\mathrm{sp}(A)$.

From p. 318, in the proof of Theorem 5.2.8:

$$\langle Ax, x\rangle = \int_{-\|A\|}^{\|A\|} \lambda \,\mathrm{d}\langle E_\lambda x, x\rangle \quad \text{for all } x \in \mathscr{H}.$$

We want to show that $AE_\lambda \leq \lambda E_\lambda$ and $\lambda(I - E_\lambda) \leq A(I - E_\lambda)$. Recall,

$$\langle E_{\lambda'} x, x\rangle - \langle E_\lambda x, x\rangle = \mu_x(\lambda, \lambda'],$$

and integration with respect to this measure μ_x is often denoted $\mathrm{d}\langle E_\lambda x, x\rangle$. By the linearity and continuity of the inner product in the first variable, we have

$$\langle Ax, x\rangle = \int_{-\|A\|}^{\|A\|} \lambda \,\mathrm{d}\langle E_\lambda x, x\rangle \;= \Big\langle \Big(\int_{-\|A\|}^{\|A\|} \lambda \,\mathrm{d}E_\lambda\Big) x, x\Big\rangle$$

for all $x \in \mathscr{H}$, whence $A = \int_{-\|A\|}^{\|A\|} \lambda \,\mathrm{d}E_\lambda$, by Proposition 2.4.3. Arguing as in the second paragraph of the proof of Theorem 5.2.3, we obtain the desired inequalities.

Remark on Theorem 5.2.8 (Bounded Borel function calculus): The operator $g(A)$ is really determined entirely by the values g takes on $\mathrm{sp}(A)$ (for this see Theorem 5.2.8 and its proof, as well as the paragraph following Theorem 5.2.8). This is because the measures $\mu_{x,y}$, which come from the measures μ_x, are supported on $\mathrm{sp}(A)$, as discussed on p. 318.

We know that $g(A)$ will depend only on the values of g on the spectrum, so functions on $\mathbb{C}$ or X will determine the same operator $g(A)$. Thus, we may extend any bounded Borel function on $\mathbb{C}$ to a bounded Borel function on X by simply assigning a value to the point at infinity, and enlarging the domain in this way will not alter the operator $g(A)$.

The idea of Theorem 5.2.8 is to extend the representation of the C^*-algebra $C(X)$ on $\mathscr{H}$, whose image is an abelian C^*-algebra $\mathfrak{U}(A)$, which was given by Theorem 4.4.5, to a representation of $\mathscr{B}$ into the double-commutant of $\mathfrak{U}(A)$, where $\mathscr{B}$ denotes the bounded Borel functions on $\mathbb{C}$. We recall that here $X = \{\mathbb{C}, \infty\}$ denotes the Riemann sphere, or $X = \{\mathbb{R}, \infty\}$, according as A is normal or self-adjoint. This first appearance in the argument of the extension is on p. 319, when K&R write in parentheses "this is apparent from (3) when $g \in C(X)$, and extends by a measure-theoretic argument to the case where g is a bounded Borel function". What is meant here is that since the measures $\mu_{x,y}$ live on $\mathrm{sp}(A)$, one can show using a measure theoretic argument that

$$\Big| \int_{\mathbb{C}} g(p) \, \mathrm{d}\mu_{x,y}(p) \Big| \leq \|x\| \|y\| \sup_{p \in \mathrm{sp}(A)} |g(p)|.$$

In the remainder of the proof K&R continue to integrate over X, while speaking of Borel functions on $\mathbb{C}$, but the brief parenthetical remark above is the key to understanding that in what follows one could just as well integrate over $\mathbb{C}$. The linear functional so obtained would be the same, because the domain of integration differs only on a set of measure zero; hence we would obtain $g(A)$ unambiguously. I find it unclear that they are not consistent on this point. It would be better simply to work consistently over $X = \{\mathbb{C}, \infty\}$, and then apply the argument near the bottom of p. 321 to extend to Borel subsets of X containing $\mathrm{sp}(A)$.

The need for *bounded* Borel functions comes up at a several points in the argument. One instance is the one just mentioned: we need $\sup_{p \in \mathrm{sp}(A)} |g(p)|$ to be finite. There are also places where we need a Weierstrass approximation argument: for example, when we prove the composite function rule, at the point where we wish to bound the range of a function h. We want to apply the Stone-Weierstrass argument to a closed disk containing the range of h. The Weierstrass approximation we want is by polynomials in the uniform norm metric, so we need a bounded subset of $\mathbb{C}$. The employment of Weierstrass approximation arguments is also why, at the top of page 322, that the text states that the statement and proof of 5.2.9 can be modified to apply to *bounded* Borel subsets of $\mathbb{C}$: but this application is "concealed" in the invocation of Theorem 4.4.5 applied to φ in the proof of 5.2.9. Essentially it is theorem 4.4.5 that can be modified to $C(\mathcal{U})$, if $\mathcal{U}$ is a bounded Borel set containing of $\mathbb{C}$ containing $\mathrm{sp}(A)$. The need for *Borel* subsets containing $\mathrm{sp}(A)$ comes from a step in the paragraph near the bottom of p. 321: to define $g(A)$ when g is defined on some bounded Borel set $\mathcal{U}$ containing $\mathrm{sp}(A)$, we first extended g to a function $\tilde{g}$ on $\mathbb{C}$ by setting it to be zero off $\mathcal{U}$; the function so obtained will be a Borel function provided that $\mathcal{U}$ is a Borel set.

We also note that at one point in the proof of the bounded Borel calculus there is a step that could use elaboration, which is pertinent to the discussion above. In the middle of page 320, where a Weierstrass approximation is used to approximate a continuous function g uniformly by polynomials, we are restricting to a closed disk $\mathbb{D} \subset \mathbb{C}$ that contains both the range of h and $\mathrm{sp}\big(h(A)\big)$—this approximation works for any fixed g and h. For fixed g and h as above, by approximating g on $\mathbb{D}$ containing range(h) and $\mathrm{sp}\big(h(A)\big)$, we are able to conclude that (8) holds for $g|_{\mathbb{D}}$ and h, *ie*

$$(g|_{\mathbb{D}} \circ h)(A) = g|_{\mathbb{D}}\big(h(A)\big).$$

Now since $g|_{\mathbb{D}} \circ h = g \circ h$ on all of $\mathbb{C}$ (and in particular, on $\mathrm{sp}(A)$), we have

$$\begin{aligned}(g \circ h)(A) &= (g|_{\mathbb{D}} \circ h)(A) \\ &= g|_{\mathbb{D}}\big(h(A)\big) \\ &= g|_{\mathrm{sp}(h(A))}\big(h(A)\big) \\ &= g\big(h(A)\big).\end{aligned}$$

Since g, h were arbitrary, we have $(g \circ h)(A) = g\big(h(A)\big)$ for all continuous g on $\mathbb{C}$ and all bounded Borel h on $\mathbb{C}$.

The same type of argument applies to Remark 5.2.15: here we have $\mathbb{D}$ a closed disk that contains the range of g and $\operatorname{sp}\big(g(A)\big)$. By deferring to 5.2.9 (modified to bounded subsets of $\mathbb{C}$ containing the spectrum), we obtain

$$(f|_{\mathbb{D}} \circ g)(A) = f|_{\mathbb{D}}\big(g(A)\big).$$

We note that we need to restrict to the disk here in order to apply 5.2.9 (which also relies, in its proof, on a Weierstrass approximation). Now it follows as in the previous paragraph that

$$(f \circ g)(A) = f\big(g(A)\big).$$

The paragraph following Theorem 5.2.8: In previous function calculi, we had the function calculus proved for functions on $C\big(\operatorname{sp}(A)\big)$, and wished to enlarge to functions defined on bigger domains (cf. the paragraph preceding Theorem 4.1.8, and the second paragraph on p. 273); this was easy, because we could simply define $f(A)$ to be $f\big|_{\operatorname{sp}(A)}(A)$. Now the situation is that we have the function calculus of Theorem 5.2.8 defined for functions on the larger domain $\mathbb{C}$, but we can—without ambiguity—shrink the domain to any Borel subset containing $\operatorname{sp}(A)$.

As noted in the text, $\mathscr{I}$ is the kernel of the homomorphism of $\mathscr{B}$ onto $\mathscr{B}\big(\operatorname{sp}(A)\big)$ given by $g \mapsto g\big|_{\operatorname{sp}(A)}$; and hence, by the 1st isomorphism theorem, $\mathscr{B}/\mathscr{I} \cong \mathscr{B}\big(\operatorname{sp}(A)\big)$. If $q : \mathscr{B} \to \mathscr{B}/\mathscr{I}$ denotes the canonical quotient map, and ρ denotes the restriction map, then the previous sentence could be expressed by saying ρ "factors through" q, and this could be represented by saying there exists $\tilde{\rho}$ such that the following triangle commutes:

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "\rho"] \arrow[d, "q"] & \mathscr{B}\big(\operatorname{sp}(A)\big) \\
\mathscr{B}/\mathscr{I} \arrow[ur, "\cong", "\tilde{\rho}"']
\end{tikzcd}$$

The authors describe a map that sends $g \in \mathscr{B}\big(\operatorname{sp}(A)\big)$ to $\tilde{g} \in \mathscr{B}$, where

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in \operatorname{sp}(A) \\ 0 & \text{otherwise;} \end{cases}$$

let's call this map η. Then $\tilde{\rho}^{-1}$ is given by $q \circ \eta$. As the authors point out, since the kernel of the map sending $g \in \mathscr{B}$ to $g(A) \in \mathscr{A}$ contains $\mathscr{I}$, this map (let's call it ϕ) factors through q as well:

$$\begin{tikzcd} & \mathscr{B} \arrow[r, "\phi"] \arrow[d, "q"] & \mathscr{A} \\ \mathscr{B}\big(\mathrm{sp}(A)\big) \arrow[r, "\cong", "\tilde{\rho}^{-1}"'] & \mathscr{B}/\mathscr{I} \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

Thus, when the authors say "the mapping, $g + \mathscr{I} \to g(A)$, gives rise to a homomorphism of $\mathscr{B}\big(\mathrm{sp}(A)\big)$ into $\mathscr{A}$," they mean the composition $\tilde{\phi} \circ \tilde{\rho}^{-1}$, in our notation.

5.2.9, proof We wish to show that the collection $\mathscr{F}$ of all Borel subsets of $\mathrm{sp}(A)$ whose characteristic functions g satisfy $\varphi(g) = g(A)$, contains the open sets and is closed under countable unions and complements, and hence coincides with the Borel σ-algebra on $\mathrm{sp}(A)$.

To show that $\varphi(\chi_{\mathcal{U}}) = \chi_{\mathcal{U}}(A)$ for each open set $\mathcal{U} \subset \mathrm{sp}(A)$, we imitate the relevant part of the proof of 5.2.8. We need continuous functions g_n that converge pointwise to g. So we express $\mathcal{U}$ as a countable union of open disks $\mathscr{O}_j$ with radius r_j. We let f_{jn} be a continuous function on $\mathrm{sp}(A)$ with range in $[0,1]$, vanishing outside $\mathscr{O}_j$, and taking the value 1 on the closed disk with the same center as $\mathscr{O}_j$ and with radius $\frac{(n-1)r_j}{n}$. (For this we use Urysohn's lemma.) Finally we take $g_n = f_{1n} \vee f_{2n} \vee \cdots \vee f_{nn}$. We know that $\varphi(g_n) = g_n(A)$ for each n, because the g_n are continuous. By σ-normality and 5.1.4, $\varphi(g_n)$ is strong operator convergent to its l.u.b. $\varphi(g)$; and likewise, $g_n(A)$ is strong operator convergent to its l.u.b. $g(A)$; since the strong operator topology is Hausdorff, $g(A) = \varphi(g)$.

We now check that $\mathscr{F}$ is closed under countable unions. Let $\{E_j\}_{j=1}^{\infty}$ be such that $E_j \in \mathscr{F}$ for each j, and let $E = \bigcup_{j=1}^{\infty} E_j$. Then the functions $g_N = \chi_{\bigcup_{j=1}^{N} E_j}$ define an increasing family of bounded Borel functions converging pointwise to $g = \chi_E$. Since φ is σ-normal, $\varphi(g_N) \to \varphi(g)$ in the strong operator sense, and likewise, by

σ-normality, $g_n(A) \to g(A)$ in the strong operator sense. But for each N,

$$\varphi(g_N) = \varphi\big(\sum_{j=1}^{N} E_j\big) = \sum_{j=1}^{N} \varphi(\chi_{E_j}) = \sum_{j=1}^{N} \chi_{E_j}(A) = \big(\sum_{j=1}^{N} \chi_{E_j}\big) = (\chi_{\bigcup_{j=1}^{N} E_j})(A) = g_N(A).$$

It follows by uniqueness of limits that $g(A) = \varphi(g)$, so that $\mathscr{F}$ is closed under countable unions.

Now if $E \in \mathscr{F}$, then

$$\varphi(\chi_{E^c}) = \varphi(1 - \chi_E) = \varphi(1) - \varphi(\chi_E) = I - \chi_E(A).$$

But $I = 1(A) = \chi_{\mathrm{sp}(A)}(A)$, so the above is equal to

$$\chi_{\mathrm{sp}(A)}(A) - \chi_E(A) = \chi_{\mathrm{sp}(A)\cap E^c}(A) = \chi_{E^c}(A).$$

Thus $\mathscr{F}$ is closed under complements.

5.2.11, proof The second sentence follows from Lemma 5.2.10.

Remark 5.2.12 Note that Lemma 5.2.10 does require $\mathrm{AC}_{\boldsymbol{\omega}}$, the axiom of countable choice. Does 5.2.5? Also, the final line of this remark "and $\exp iH = U$" follows from 4.1.8(ii).

Remark 5.2.13 Note that range$(f) = \mathrm{sp}(A)$ by the equation displayed in the fourth line from the bottom on p. 271. That $g \mapsto g \circ f$ is σ-normal follows immediately from the function composition rule for bounded Borel functions.

Remark 0.1. We make a note here that pertains to σ-normality of an abelian von Neumann algebra $\mathscr{A}$, relevant to Remark 5.2.13 in the text. We know from 4.4.3 that if $\mathscr{A}$ is an abelian von Neumann algebra, then $\mathscr{A} \cong C(X)$ for a compact Hausdorff space X, via a map ψ. We assert that the map ψ is σ-normal. If $\{f_n\} \in C(X)$, $f_n \to f$ pointwise (and hence uniformly), and $f_n \leq f_{n+1}$ for all n, then in particular, the f_n are real-valued, and f is a least upper bound for $\{f_n\}$. Thus $\psi(f_n)$ and $\psi(f)$ are self-adjoint, and $\{\psi(f_n)\}$ is an increasing sequence, as a consequence of 4.1.8(i). We also know from 4.1.8(i) that ψ is continuous, and hence $\psi(f_n) \to \psi(f)$ in norm.

This implies convergence in the strong operator topology as well, and therefore $\psi(f)$ is the least upper bound of $\{\psi(f_n)\}$ (argue as in the last part of the proof of Lemma 5.1.4).

5.2.14: there are some little steps here. It is shown in the argument that if $h \in \mathscr{B}\big(\mathrm{sp}(A)\big)$ is such that h vanishes on $\mathrm{sp}\big(\phi(A)\big)$, then $\phi\big(h(A)\big) = 0$. In the equation displayed in the middle of the page, we apply this to the function $\imath \cdot g$, which vanishes on $\mathrm{sp}\big(\phi(A)\big)$ since g does, to ascertain that $\phi\big((\imath \cdot g)(A)\big) = 0$. We then have

$$\begin{aligned}
\phi\big(\imath(A)\big) &= \phi\big(\imath(A)\big) - \phi\big((\imath \cdot g)(A)\big) \\
&= \phi\big(\imath(A) - (\imath \cdot g)(A)\big) \\
&= \phi\big((\imath - \imath \cdot g)(A)\big) \\
&= \phi\big([\imath \cdot (1 - g)](A)\big).
\end{aligned}$$