Notes for Takesaki Vol. 1

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p. 67 "Since $\mathscr{L}(\mathfrak{H})$ is the conjugate space of the Banach space $\mathscr{L}_*(\mathfrak{H})$, we can naturally define the $\sigma\big(\mathscr{L}(\mathfrak{H}), \mathscr{L}_*(\mathfrak{H})\big)$-topology." In other words, since $\mathscr{L}(\mathfrak{H}) = \mathscr{L}_*(\mathfrak{H})^*$, we can define the $\sigma\big(\mathscr{L}_*(\mathfrak{H})^*, \mathscr{L}_*(\mathfrak{H})\big)$-topology: this is the weakest topology on $\mathscr{L}_*(\mathfrak{H})^*$ such that each element of $\mathscr{L}_*(\mathfrak{H})$, regarded as a linear functional on $\mathscr{L}_*(\mathfrak{H})^*$, is continuous. In other words, it is the weakest topology such that each element of $\mathscr{L}_*(\mathfrak{H})$, regarded as an evaluation map on $\mathscr{L}_*(\mathfrak{H})^*$, is continuous.

This can be generalized as follows: for a Banach space E, $\sigma(E^*, E)$ is the weakest topology on E^* such that all the maps $\{\mathrm{ev}_x : x \in E\}$ are continuous on E^*: *ie* $\sigma(E^*, E)$ is the weak-* topology on E^*. So the $\sigma\big(\mathscr{L}(\mathfrak{H}), \mathscr{L}_*(\mathfrak{H})\big)$-topology (which is to say the $\sigma\big(\mathscr{L}_*(\mathfrak{H})^*, \mathscr{L}_*(\mathfrak{H})\big)$-topology), is the weak-* topology on $\mathscr{L}_*(\mathfrak{H})^*$. In other words, the σ-weak topology on $\mathscr{L}(\mathfrak{H})$ is the weak-* topology on $\mathscr{L}_*(\mathfrak{H})^* = \mathscr{L}(\mathfrak{H})$.

p. 124 Second-last paragraph: "with respect to the polar considered in the universal enveloping von Neumann algebra $\tilde{A}$ and in A^*, respectively". This is backwards, it should be the other way around if we are to be consistent with how the term *polar* is used at the top of the page. It should read "considered in A^* and in the universal enveloping von Neumann algebra $\tilde{A}$, respectively". Also, in the same paragraph there is a typo: it should read "by means of".

p. 253 Section 7: for the use of *locally ae* and *locally null* with regard to L^p and L^∞ in this section, see Hewitt and Ross, vol. 1, §12, as well as parts of §11. Without being familiar with that material, this section of Takesaki is nearly impossible to understand.

p. 256 "Conversely, suppose f is an E-valued μ-measurable function with $\|f\|_p < +\infty$. It follows then that there exists a disjoint sequence $\{K_n\}$ of compact subsets of Γ such that $f(\gamma) = 0$ locally μ-almost everywhere in $(\bigcup_{n=1}^\infty K_n)^c$."

This uses Theorem (11.39) on p. 129 of Hewitt and Ross, vol. 1.

p. 259 "Each $\pi(f)$, $f \in L^\infty(\Gamma, \mu)$, given by (3) is, of course, decomposable". For this we define a bounded operator $M_f : \Gamma \to \mathcal{L}(\mathfrak{H})$ by $[M_f(\gamma)](\eta) = f(\gamma)\eta$ for each

$\eta \in \mathfrak{H}$. Then by (3) we have

$$M_f(\gamma)\xi(\gamma) = f(\gamma)\xi(\gamma) = (\pi(f)\xi)(\gamma),$$

and thus (4) holds.

p. 259 "It is clear that every decomposable operator commutes with $\mathscr{A}$". We verify this easy step. Let $x \in \mathcal{L}\big(L^2_{\mathfrak{H}}(\Gamma,\mu)\big)$ be decomposable, and let $\pi(f) \in \mathscr{A}$. Then there is a bounded, measurable, $\mathcal{L}(\mathfrak{H})$-valued function on Γ that satisfies

$$(x\xi)(\gamma) = x(\gamma)\xi(\gamma)$$

for each $\xi \in L^2_{\mathfrak{H}}(\Gamma,\mu)$. Let $\xi \in L^2_{\mathfrak{H}}(\Gamma,\mu)$. For each $\gamma \in \Gamma$ we have

$$\begin{aligned}
[\big(x\pi(f)\big)\xi](\gamma) &= [x\big(\pi(f)\xi\big)](\gamma) \\
&= x(\gamma)\big(\pi(f)\xi\big)(\gamma) \\
&= x(\gamma)\big(\underbrace{f(\gamma)}_{\in\mathbb{C}}\xi(\gamma)\big) \\
&= f(\gamma)x(\gamma)\big(\xi(\gamma)\big) \\
&= f(\gamma)(x\xi)(\gamma) \\
&= \big(\pi(f)(x\xi)\big)(\gamma) \\
&= [\big(\pi(f)x\big)\xi](\gamma),
\end{aligned}$$

whence $x\pi(f)\xi = \pi(f)x\xi$ for each $\xi \in L^2_{\mathfrak{H}}(\Gamma,\mu)$. Thus $x\pi(f) = \pi(f)x$.

p. 260 "By definition, it follows that for every $\xi \in \mathfrak{H}$, we have $x(\gamma)\xi = y(\gamma)\xi$ locally μ=almost everywhere." We verify this small step. Given $x(\cdot), y(\cdot) : \Gamma \to \mathcal{L}(\mathfrak{H})$ bounded, measurable functions such that

$$\int_\Gamma^\oplus x(\gamma)\,\mathrm{d}\mu(\gamma) = \int_\Gamma^\oplus y(\gamma)\,\mathrm{d}\mu(\gamma).$$

In other words, we have

$$(x\xi)(\gamma) = (y\xi)(\gamma) \quad \text{for all } \xi \in L^2_{\mathfrak{H}}(\Gamma,\mu),$$

ie,

$$x(\gamma)\xi(\gamma) = y(\gamma)\xi(\gamma) \quad \text{for all } \xi \in L^2_{\mathfrak{H}}(\Gamma, \mu).$$

Given $\eta \in \mathfrak{H}$, for any compact neighbourhood $\mathcal{U} \subset \Gamma$, if we let $\xi = \mathbb{1}_{\mathcal{U}}\eta$ then $\xi(\gamma) = \eta$ on $\mathcal{U}$.

p. 264 The grammar is awkward in one spot. I would say "Borel space generated by the topology of a Polish space" rather than "the Borel space of a Polish space generated by the topology".

Also, because Takesaki uses the term Borel structure to mean σ-algebra, the definition could be confusing to the reader unfamiliar with this terminology. Really the Effros Borel structure is just the smallest σ-algebra on $\mathfrak{W}(E^*)$ making the maps $\psi : F \in \mathfrak{W}(E^*) \mapsto \|x|_F\|$ measurable; *ie* it is the σ-algebra generated by the collection of sets

$$\{\psi^{-1}(\mathcal{U}) : \mathcal{U} \subset \mathbb{R} \text{ is open}\}.$$

p. 268–269 In the statements of Corollaries 8.5 and 8.7, we need the fact that a measurable subset of a standard Borel space is itself a standard Borel space, when regarded as a measurable subspace. See [2] for more information on Standard Borel spaces.

p. 272 Line 1: Correction: isometry of $\mathfrak{H}(\gamma)$ into $\mathfrak{H}_0$.

p. 273 "Effros Borel structure in the space $\mathfrak{A}(\mathfrak{H}_n)$ of all von Neumann algebras on $\mathfrak{H}_n$" This means to consider the relative σ-algebra on $\mathfrak{A}(\mathfrak{H}_n)$ inherited from the Effros Borel σ-algebra on $\mathfrak{W}(\mathscr{L}_*(\mathscr{H})^*) = \mathfrak{W}(\mathscr{L}(\mathscr{H}))$, *ie*, to consider $\mathfrak{A}(\mathfrak{H}_n)$ as a measurable subspace of $\mathfrak{W}(\mathscr{L}_*(\mathscr{H})^*)$. See [1] for a discussion of the concept of measurable subspace and for the terminology "Borel space"; Takesaki uses the term Borel space to mean what is usually called a measurable space, and uses the term Borel structure for what is usually called a σ-algebra.

So to consider the collection of all von Neumann algebras on $\mathscr{H}$ with the Effros Borel structure we must verify that $\mathfrak{A}(\mathscr{H}) \subset \mathfrak{W}(\mathscr{L}_*(\mathscr{H})^*)$, *ie* we must verify that each von Neumann algebra on $\mathscr{H}$ is indeed a weak-* closed subset of $\mathscr{L}_*(\mathscr{H})^*$. This is not hard, because if M is a von Neumann algebra on $\mathscr{H}$, then by definition it is weak-operator closed in $\mathscr{L}(\mathscr{H})$; it follows from the results on p. 68 that M is σ-weakly closed in $\mathscr{L}(\mathscr{H})$, which is equivalent to its being weak-* closed in $\mathscr{L}_*(\mathscr{H})^*$.

References

[1] Measurable space. *Encyclopedia of Mathematics.* URL: `http://www.encyclopediaofmath.org/index.php?title=Measurable_space&oldid=28120`

[2] Standard Borel space. *Encyclopedia of Mathematics.* `http://www.encyclopediaofmath.org/index.php?title=Standard_Borel_space&oldid=43097`