Path-connectedness of superlevel sets of eigengap functions in the space of Hermitian matrices

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Abstract

In this paper we show that the superlevel sets of the eigengap functions in the space of Hermitian matrices are smoothly path-connected. We do this by choosing two arbitrary matrices in the superlevel set of the kth eigengap function, and constructing a smooth path between them.

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While properties of eigenvalues of Hermitian matrices have long been studied, the topology of the space of Hermitian matrices having distinct eigenvalues does not appear to be well-known. In what follows we attempt to shed a bit of light on this problem by showing that the space of Hermitian matrices with a pair of successive eigenvalues separated by some fixed distance is path-connected. We begin with a couple definitions.

Definition 0.1. For any Hermitian matrix H with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, we define its kth *eigengap* to be

$$\Delta_k(H) = \lambda_{k+1} - \lambda_k.$$

We refer to Δ_k as the kth *eigengap function.*

Definition 0.2. Let $\mathcal{H}$ be the space of Hermitian matrices with the relative topology from $\mathrm{GL}(n)$. Let $\mathcal{H}_k^c = \{H \in \mathcal{H} : \Delta_k(H) \geq c\}$, where $c \in [0, \infty)$. That is, let $\mathcal{H}_k^c$ be the superlevel set of Δ_k above c, considered as a subset of $\mathcal{H}$.

Theorem 0.3. $\mathcal{H}_k^c$ is smoothly path-connected for every $c \geq 0$, and each $k = 1, 2, \ldots, n$.

Proof. We invoke the finite-dimensional Spectral Theorem, which says that any normal matrix M is unitarily diagonalizable, that is, $M = PDP^*$, where D is the diagonal matrix of eigenvalues of M, and P is the unitary matrix whose columns are the corresponding orthonormal eigenvectors. We can rearrange the order in which the eigenvalues appear in D by choosing a different change-of-basis matrix, whose columns are still the eigenvectors of M, but arranged in a different order (the jth column corresponds to the eigenvalue which is the jth diagonal element of D). Doing so does not change the fact that the change-of-basis matrix is unitary, since an equivalent condition for a matrix to be unitary is for its columns to be mutually orthonormal.

Let A and B be in $\mathcal{H}_k^c$, with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the eigenvalues of A, and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ the eigenvalues of B. Let $D_1 = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$; $D_2 = \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_n\}$. There exist P and Q so that $A = PD_1P^*$ and $B = QD_2Q^*$.

Now let $\phi(t) = (1-t)D_1 + tD_2$, for $t \in [0,1]$. Explicitly,

$$\phi(t) = \begin{pmatrix} (1-t)\lambda_1 + t\mu_1 & & & \\ & (1-t)\lambda_2 + t\mu_2 & & \\ & & \ddots & \\ & & & (1-t)\lambda_n + t\mu_n \end{pmatrix}.$$

We make a few observations about ϕ. That $\Delta_k(\phi)$ is greater than or equal to c follows easily from the fact that $\Delta_k(A)$ and $\Delta_k(B)$ are:

$$\begin{aligned} \Delta_k(\phi(t)) &= (1-t)\lambda_{k+1} + t\mu_{k+1} - [(1-t)\lambda_k + t\mu_k] \\ &= (1-t)(\lambda_{k+1} - \lambda_k) + t(\mu_{k+1} - \mu_k) \\ &\geq (1-t)c + tc \\ &= c. \end{aligned}$$

Lastly, since the conjugate transpose of any diagonal matrix with real entries is itself, $\phi(t)$ is Hermitian for all t.

We now let Φ be a path in the unitary group $\mathrm{U}(n)$ from P to Q. Consider $\Psi = \Phi\phi\Phi^*$. Since Φ and ϕ are smooth paths, and matrix multiplication and inverse are smooth operations in $\mathrm{GL}(n)$, we see that Ψ is a smooth path in $\mathcal{H}_k^c$ from A to B. Since similarity preserves eigenvalues, Ψ will have the same eigenvalues as ϕ, and hence the same eigengaps. Finally, $\Psi(t)^* = (\Phi(t)\phi(t)\Phi(t)^*)^* = \Phi(t)^{**}\phi(t)^*\Phi(t)^* = \Phi(t)\phi(t)\Phi(t)^* = \Psi(t)$, so that $\Psi(t)$ is Hermitian for all t. $\square$

Remark 0.4. The inequality " $\geq$ " may be replaced with " $\leq$ " or " $=$ " to show that the sublevel sets $\{H \in \mathcal{H} : \Delta_k(H) \leq c\}$ and the level sets $\{H \in \mathcal{H} : \Delta_k(H) = c\}$ are smoothly path-connected. Or we may replace it with " $>$ " or " $<$ " to obtain that the sets $\{H \in \mathcal{H} : \Delta_k(H) > c\}$ and $\{H \in \mathcal{H} : \Delta_k(H) < c\}$ are smoothly path-connected.

Remark 0.5. The analogous result for real symmetric matrices may not hold, since the orthogonal group over $\mathbb{R}$ is not connected, but rather has two connected components. However, if we let $\mathcal{H}_k^c+$ be the intersection of $\mathcal{H}_k^c$ with the set of Hermitian matrices with change-of-basis matrix in the unitary diagonalization having determinant 1, then $\mathcal{H}_k^c+$ is path-connected. Similarly for the intersection of $\mathcal{H}_k^c$ with the set of Hermitian matrices with change-of-basis matrix in the unitary diagonalization having determinant -1. This means that in the case of real symmetric matrices, the superlevel (and level and sublevel) sets have at most two path-components.