Notes and calculations for Keith Taylor's C^*-Algebras of Crystal Groups

(in progress)

Tom Potter

We first verify that with regard to the action of D on A we have

$$\boxed{d^{-1} \cdot a = \gamma(d)^{-1} a \gamma(d)} \tag{1}$$

for any $d \in D$. Recall that the action of D on A is given by $d \cdot a = \gamma(d) a \gamma(d)^{-1}$. It follows from this that

$$d^{-1} \cdot a = \gamma(d^{-1}) a \gamma(d^{-1})^{-1}.$$

Note that

$$\gamma(d)\gamma(d^{-1}) = \gamma(dd^{-1})\alpha(d, d^{-1}) = \gamma(e)\alpha(d, d^{-1}) = e_G \alpha(d, d^{-1}) = \alpha(d, d^{-1}),$$

where e_G denotes the identity in G. It follows that

$$\gamma(d^{-1}) = \gamma(d)^{-1} \alpha(d, d^{-1}),$$

whence

$$\boxed{\gamma\big(d^{-1}\big)^{-1} = \alpha\big(d, d^{-1}\big)^{-1} \gamma(d)} \tag{2}$$

for all $d \in D$. The above equation will be used frequently. With this equation we see that

$$\begin{aligned} d^{-1} \cdot a &= \gamma(d^{-1}) a \gamma(d^{-1})^{-1} \\ &= \gamma(d)^{-1} \alpha(d, d^{-1}) a \alpha\big(d, d^{-1}\big)^{-1} \gamma(d) \\ &= \gamma(d)^{-1} \alpha(d, d^{-1}) \alpha\big(d, d^{-1}\big)^{-1} a \gamma(d) \\ &= \gamma(d)^{-1} a \gamma(d), \end{aligned}$$

where we have used the fact that A is abelian.

We also record for future reference that

$$\boxed{\alpha(b, e_D) = \alpha(e_D, b) = e_G \quad \text{for all } b \in D,}$$

where e_D denotes the identity of D; this follows trivially from the fact that $\gamma(e_D) = e_G$.

Page 513, first and fourth paragraphs; and Page 517, second paragraph

Let $N \subset G$ be a normal subgroup of finite index. On page 517 we have an isomorphism $U : L^2(G) \to \bigoplus_{d\in D} L^2(N)$ by $(Ug)_d(t) = g(\gamma(d)t)$ for $t \in N, d \in D$, and $g \in L^2(G)$. This map is well-defined because $G = \dot{\bigcup}_{d\in D}\gamma(d)N$. This choice of notation is not optimal since U has been used elsewhere, so we will instead call this map θ.

So we have the isomorphism $\theta : L^2(G) \to \bigoplus_{d\in D} L^2(N)$ where $\theta g = \big((\theta g)_d\big)_{d\in D}$ is given by $(\theta g)_d(t) = g\big(\gamma(d)t\big)$. In the paper the map θ is almost always suppressed by using the notation $\theta g = (g_d)_{d\in D}$ where g_d is defined by $g_d(t) = g\big(\gamma(d)t\big)$; in other words, we write $(\theta g)_d(t) = g_d(t)$; we will suppress the θ in these notes as well. We note that we can make the same definition for $g \in L^p(G)$ for other p values, and

Proposition 0.1. For $1 \leq p \leq \infty$,

$$\boxed{g \in L^p(G) \iff g_d \in L^p(N) \text{ for all } d}.$$

We will have more to say about the isomorphism θ presently, and we will prove the above easy proposition. We note in passing that

$$\boxed{g\big(\gamma(d)t\big) = g(d, t) = g_d(t)}$$

in the notation of the paper, for $g : G \to \mathbb{C}$; there is, however, a minor notational inconsistency in the third paragraph of page 513, of which the reader should be careful.

We note that for any $g \in L^2(G)$ we have $g = \sum_{c\in D} \widetilde{g_c}$, where $\widetilde{g_c} \in L^2(G)$ is given by

$$\widetilde{g_c}(t) = \left\{ \begin{array}{ll} g(t) & \text{if } t \in \gamma(c)N \\ 0 & \text{otherwise} \end{array} \right\} = g(t)\mathbb{1}_{\gamma(c)N}(t)$$

for all $g \in G$. We note that $\widetilde{g_c}$ is not to be confused with g_c; however, the two are related: we have, for any $n \in N$,

$$\begin{aligned}\widetilde{g_b}\big(\gamma(c)n\big) &= \left\{\begin{array}{ll} g\big(\gamma(c)n\big) & \text{if } c = b \\ 0 & \text{otherwise} \end{array}\right\} \\ &= \left\{\begin{array}{ll} g_c(n) & \text{if } c = b \\ 0 & \text{otherwise} \end{array}\right\} \\ &= g_b(n)\delta_{bc}.\end{aligned}$$

Hence $(\widetilde{g_b})_c(n) = g_b(n)\delta_{bc}$ for all $n \in N$; *ie*,

$$\boxed{(\widetilde{g_b})_c = g_b\delta_{bc}}. \tag{3}$$

We now prove the simple proposition given above: suppose first that $1 \le p < \infty$; if $g \in L^p(G)$ then $g_d(t) = g\big(\gamma(d)t\big)$, and hence

$$\int_N |g_d(t)|^p \, \mathrm{d}t = \int_N |g\big(\gamma(d)t\big)|^p \, \mathrm{d}t \le \int_G |g\big(\gamma(d)t\big)|^p \, \mathrm{d}t < \infty,$$

and therefore $g_d \in L^p(N)$ for each d. Conversely, if $g_d \in L^p(N)$ for each $d \in D$, then

$$\int_G |g(t)|^p \, \mathrm{d}t = \int_G \Big|\sum_{c\in D} \widetilde{g_c}(t)\Big|^p \, \mathrm{d}t = \int_G \Big|\sum_{c\in D} g(t)\mathbb{1}_{\gamma(c)N}(t)\Big|^p \, \mathrm{d}t = \int_G \sum_{c\in D} |g(t)|^p \mathbb{1}_{\gamma(c)N}(t) \, \mathrm{d}t.$$

By linearity of the integral this becomes

$$\sum_{c\in D} \int_{\gamma(c)N} |g(t)|^p \, \mathrm{d}t = \sum_{c\in D} \int_N |g\big(\gamma(c)t\big)|^p \, \mathrm{d}t = \sum_{c\in D} \int_N |g_c(t)|^p \, \mathrm{d}t < \infty.$$

Note that this works because N is of finite index in G. This proves our little proposition for $1 \le p < \infty$, and the case $p = \infty$ is trivial. *In these notes we are only interested in* $p = 1, 2$.

Remark 0.2. We are now in a position to describe more explicitly the inverse of the map θ discussed earlier. We claim that for $(g_d)_{d\in D} \in \mathcal{H} = \bigoplus_{d\in D} L^2(N)$,

$$\boxed{\theta^{-1}\big[(g_d)_{d\in D}\big] = \sum_{d\in D} \widetilde{g_d} \in L^2(G)}, \tag{4}$$

where $\widetilde{g_d}$ is as above with g is defined by $g\big(\gamma(d)t\big) = g_d(t)$. Since θ is easily seen to be a linear bijection, it does indeed have an inverse. And

$$\Big[\theta\Big(\sum_{c\in D}\widetilde{g_c}\Big)\Big]_d(t) = \Big[\sum_{c\in D}\theta\widetilde{g_c}\Big]_d(t) = \sum_{c\in D}\big(\theta\widetilde{g_c}\big)_d(t)$$

for all $t \in N$, and recalling our notational suppression of θ this becomes

$$\sum_{c\in D}(\widetilde{g_c})_d(t) = \sum_{c\in D} g_c(t)\delta_{cd} = g_d(t) = \big(\theta g\big)_d(t).$$

Hence $\theta\Big(\sum_{c\in D}\widetilde{g_c}\Big) = \theta(g) = (g_d)_{d\in D}$ and therefore formula (4) is justified.

We can analyze θ in even greater detail by expressing it as a composition: first define, for any $d \in D$,

$$L^2_d(N) = \big\{g \in L^2(G) : \operatorname{supp}(g) \subset \gamma(d)N\big\}.$$

We denote the map $g \mapsto \big(\theta g\big)_d$ from $L^2(G)$ to $L^2(N)$ by θ_d. Then it's easy to see that $\theta_d\big|_{L^2_d(G)}$ is an isomorphism from $L^2_d(G)$ onto $L^2(N)$. With the natural inner product on $L^2(G)$, we can recognize it as the internal direct sum of the mutually orthogonal subspaces $L^2_d(G)$, *ie* the space of all sums $\{\sum_{d\in D}\widetilde{g_d} : \widetilde{g_d} \in L^2_d(G)\}$, which is just the closed linear span $\bigvee_{d\in D} L^2_d(G)$. We know from Hilbert space theory that each $g \in L^2(G)$ can be written uniquely in this form. We also know that this internal direct sum is isomorphic to the external direct sum $\bigoplus_{d\in D} L^2_d(G) = \{(\widetilde{g_d})_{d\in D} : \widetilde{g_d} \in L^2_d(G)\}$. Let's call this isomorphism δ; this is just the map which sends $\sum_{d\in D}\widetilde{g_d}$ to $(\widetilde{g_d})_{d\in D}$. Then θ is the composition

$$L^2(G) = \bigvee_{d\in D} L^2_d(G) \xrightarrow{\delta} \bigoplus_{d\in D} L^2_d(G) \xrightarrow{\bigoplus \theta_d|_{L^2_d(G)}} \bigoplus_{d\in D} L^2(N).$$

We see from this that

$$\boxed{\theta\widetilde{g_d} = (g_c\delta_{cd})_{c\in D} \in \bigoplus_{d\in D} L^2(N)}\ ,$$

which recaptures (3) and, by linearity of θ, gives another proof of (4). For our purposes it is helpful to have all these maps explicitly at hand, so that we can use them with confidence in doing calculations.

Returning to page 513 of the paper, we recall the map in the paper given by $\Psi g = \big(\widehat{g}_d\big)_{d\in D}$. This can be written as the composition

$$\begin{tikzcd} g \arrow[r, "\theta"] \arrow[rr, bend left, "\Psi"] & (g_d)_{d\in D} \arrow[r, "\bigoplus \mathcal{P}"] & (\widehat{g}_d)_{d\in D} \end{tikzcd}$$

where $\mathcal{P} : L^2(A) \to L^2(\hat{A})$ is the Fourier transform, where $A \subset G$ is an abelian normal subgroup of finite index. In the notation established, $(\Psi g)_d = \widehat{g}_d$.

Page 513 lines 13-14: Up to isomorphism $G = \{(d, a) : d \in D, a \in A\}$ with group product $(b, a)(c, a') = (bc, \alpha(b, c)(c^{-1} \cdot a)a')$.

Remark 0.3. First note that this product is *almost* a semidirect product—it is off by a cocycle factor; it could be called a “quasi-semidirect product”. We can write the product in the notation

$$\big(\gamma(b)a\big)\big(\gamma(c)a'\big) = \gamma(bc)\alpha(b, c)(c^{-1} \cdot a)a',$$

and then for any $\gamma(b)a \in G$ we have

$$\big(\gamma(b)a\big)^{-1} = a^{-1}\gamma(b)^{-1}. \tag{5a}$$

If we wish, however, to write the inverse in the notation of the quasi-semidirect product we can write

$$(b, a)^{-1} = \big(b^{-1}, [\gamma(b)a\gamma(b^{-1})]^{-1}\big). \tag{5b}$$

Equations (5) still hold when A is not abelian, but the original group product formula must be written in the less concise form

$$(b, a)(c, a') = \big(bc, \alpha(b, c)\big(\gamma(c)^{-1}a\gamma(c)\big)a'\big)$$

in this case.

Page 513, lines 26–28: For $F = (F_{b,c})_{b,c\in D}$ in $M_n\big(C_0(\hat{A})\big)$ define $\mathcal{M}(F)$ in $\mathcal{B}(\mathcal{H})$ by for $\underline{h} = (h_c)_{c\in D}$, $\big(\mathcal{M}(F)\underline{h}\big)_b = \sum_{c\in D} F_{b,c}h_c$. Then $\mathcal{M}$ is a C^*-isomorphism of $M_n\big(C_0(\hat{A})\big)$ into $\mathcal{B}(\mathcal{H})$.

Check: We use slightly different notation. To see that $\mathcal{M}$ is linear is trivial. That $\mathcal{M}$ is injective is also easy: if $\mathcal{M}(F) = 0$, that means $\mathcal{M}(F)h = 0$ for all $h \in \mathcal{H} = \bigoplus_{c\in D} L^2(\hat{A})$, *ie*

$$\sum_{c\in D} F_{b,c}h_c = 0 \text{ for all } h \in \mathcal{H} \text{ and all } b \in D.$$

We can choose h such that $h_d \in L^2(\hat{A})$ and h_d is nonzero everywhere on $\hat{A}$, and $h_c = 0$ if $c \neq d$. Then it follows from the above that $F_{b,d} = 0$ for all $b \in D$. Since we can do this for any $d \in D$ it follows that all the components of F are 0, *ie* $F = 0$.

We check that $\mathcal{M}$ is multiplicative. We have

$$\begin{aligned}
\Big(\mathcal{M}(F)\big(\mathcal{M}(G)h\big)\Big)_d &= \sum_{b\in D} F_{d,b}\big(\mathcal{M}(G)h\big)_b \\
&= \sum_{b\in D} F_{d,b}\Big(\sum_{c\in d} G_{b,c}h_c\Big) \\
&= \sum_{b\in D}\sum_{c\in D} F_{d,b}G_{b,c}h_c \\
&= \sum_{c\in D}\sum_{b\in D} F_{d,b}G_{b,c}h_c \\
&= \sum_{c\in D}(FG)_{d,c}h_c \\
&= \big(\mathcal{M}(FG)h\big)_d.
\end{aligned}$$

We check lastly that $\mathcal{M}$ is involutive. If $F \in M_n\big(C_0(\hat{A})\big)$ then we can write F as a linear combination of its real and imaginary parts: $F = \mathrm{Re}F + \imath\mathrm{Im}F$, whence $\overline{F} = \mathrm{Re}F - \imath\mathrm{Im}F$, where $\mathrm{Re}F$ and $\mathrm{Im}F$ are self-adjoint. We can also write

$$\mathcal{M}(F) = \mathrm{Re}\big(\mathcal{M}(F)\big) + \imath\mathrm{Im}\big(\mathcal{M}(F)\big),$$

because $\mathcal{H}$ is a complex Hilbert space. If we can show that $\mathcal{M}$ takes self-adjoint elements of $M_n\big(C_0(\hat{A})\big)$ to self-adjoint elements of $\mathcal{B}(\mathcal{H})$, then it follows from linearity of

$\mathcal{M}$ that $\mathrm{Re}\big(\mathcal{M}(F)\big) = \mathcal{M}(\mathrm{Re}F)$ and $\mathrm{Im}\big(\mathcal{M}(F)\big) = \mathcal{M}(\mathrm{Im}F)$, because a bounded linear operator can be written uniquely as a linear combination of self-adjoint bounded linear operators. Hence $\mathcal{M}(\overline{F}) = \overline{\mathcal{M}(F)}$, where

$$\overline{\mathcal{M}(F)} = \mathrm{Re}\big(\mathcal{M}(F)\big) - \imath \mathrm{Im}\big(\mathcal{M}(F)\big).$$

So let $G \in M_n\big(C_0(\hat{A})\big)$ be a real-valued symmetric matrix, *ie* let G be self-adjoint in $M_n\big(C_0(\hat{A})\big)$. We check that $\mathcal{M}(G)$ is self-adjoint in $\mathcal{B}(\mathcal{H})$. Let $g, h \in \mathcal{H}$. We have

$$\begin{aligned}
\big\langle \mathcal{M}(G)g, h \big\rangle &= \sum_{d \in D} \big\langle \big(\mathcal{M}(G)g\big)_d, h_d \big\rangle \\
&= \sum_{d \in D} \Big\langle \sum_{c \in D} G_{d,c} g_c, h_d \Big\rangle \\
&= \sum_{d \in D} \sum_{c \in D} \langle G_{d,c} g_c, h_d \rangle.
\end{aligned}$$

Now switching the order of summation and using the fact that G is real and symmetric this becomes

$$\begin{aligned}
&\sum_{c \in D} \sum_{d \in D} \langle g_c, G_{c,d} h_d \rangle \\
&= \sum_{c \in D} \Big\langle g_c, \sum_{d \in D} G_{c,d} h_d \Big\rangle \\
&= \sum_{c \in D} \big\langle g_c, \big(\mathcal{M}(G)h\big)_c \big\rangle \\
&= \big\langle g, \mathcal{M}(G)h \big\rangle
\end{aligned}$$

Thus $\mathcal{M}(G)$ is self-adjoint whenever G is, and the argument made above verifies that $\mathcal{M}$ is involutive, *ie* is a $*$-homomorphism.

NOTE: *The definition of $\mathcal{M}$ really just says that $\mathcal{M}(F)$ is the element of $\mathcal{B}\big(\bigoplus_{c \in D} L^2(\hat{A})\big)$ whose matrix components are the entries of the matrix F. See §2.6 of Kadison and Ringrose for a good discussion on matrix components of operators on a direct sum. Many of the calculations shown here can be presented more cleanly (fewer sums) using matrix components, and some checking is rendered unnecessary (eg., checking that $\mathcal{M}$ is multiplicative).*

Page 513, Proposition 1: For each $f \in L^1(G)$, $\Psi\lambda_f^G\Psi^{-1}$ is in the range of $\mathcal{M}$. Let $\mathcal{F}(f) = \mathcal{M}^{-1}(\Psi\lambda_f^G\Psi^{-1})$. Then $\mathcal{F}$ extends to a C^*-isomorphism of $C^*(G)$ onto a C^*-subalgebra of $M_n\big(C_0(\hat{A})\big)$.

Check. We consider the second statement of the theorem, which follows from the first, which we will assume for now. Recall that

$$C_\lambda^*(G) = \overline{\lambda^G\big(L^1(G)\big)}^{\|\cdot\|}$$

where the closure is taken in $\mathcal{B}\big(L^2(G)\big)$, and that $C_\lambda^*(G) \cong C^*(G)$ since G is amenable. We know that $\Psi\lambda^G\big(L^1(G)\big)\Psi^{-1}$ is contained in the C^*-algebra $\mathcal{M}\big(M_n\big(C_0(\hat{A})\big)\big) \subset \mathcal{B}(\mathcal{H})$ and that $\Psi\lambda_f^G\Psi^{-1}$ is a unitary operator in $\mathcal{B}(\mathcal{H})$ for each $f \in L^1(G)$ (the details are routine here). Note that here the map $\mathcal{F}$ does not itself extend to a C^*-isomorphism, since the closure of $L^1(G)$ is not a C^*-algebra; rather, what is meant is that the map $\phi : \lambda^G\big(L^1(G)\big) \to M_n\big(C_0(\hat{A})\big)$ given by $\lambda^G(f) \mapsto \mathcal{M}^{-1}\big(\Psi\lambda^G(f)\Psi^{-1}\big)$ extends to a C^*-isomorphism from $C_\lambda^*(G)$ onto a subalgebra of $M_n\big(C_0(\hat{A})\big)$. Since $\mathcal{M}$ is a C^*-isomorphism it suffices to check that the convolution map $\phi : T \to \Psi T\Psi^{-1}$ on $\mathcal{B}(\mathcal{H})$ is a C^*-isomorphism, *ie*, that it is linear, multiplicative, isometric, and involutive. Linearity follows simply from the linearity of Ψ. Multiplicativity is also easy:

$$\phi(AB) = \Psi gh\Psi^{-1} = \Psi A\Psi^{-1}\Psi B\Psi^{-1} = \phi(A)\phi(B).$$

To see that ϕ is isometric we use the fact that Ψ is an isometric isomorphism to get

$$\begin{aligned}
\|\phi(T)\| &= \|\Psi T\Psi^{-1}\| \\
&= \|T\Psi^{-1}\| \\
&= \sup\{\|T\Psi^{-1}h\| : h \in \mathcal{H}, \|h\| = 1\} \\
&= \sup\{\|Tf\| : f \in L^2(G), \|f\| = 1\} \\
&= \|T\|,
\end{aligned}$$

whence ϕ is isometric.

To see that ϕ is involutive, we write T as $\mathrm{Re}T + \imath\mathrm{Im}T$, where $\mathrm{Re}T$ and $\mathrm{Im}T$ are self-adjoint, so that $T^* = \mathrm{Re}T - \imath\mathrm{Im}T$. We check that ϕ takes self-adjoint elements to self-adjoint elements in $\mathcal{B}(\mathcal{H})$. That ϕ is involutive will then follow from linearity of ϕ and the existence of a unique representation of elements of $\mathcal{B}(\mathcal{H})$ as a linear combination of self-adjoint elements.

Let S be self-adjoint in $\mathcal{B}(\mathcal{H})$. Then since Ψ is unitary, $\Psi^* = \Psi^{-1}$, whence

$$\phi(S)^* = (\Psi S \Psi^{-1})^* = \left(\Psi^{-1}\right)^* S^* \Psi^* = \Psi S \Psi^{-1} = \phi(S).$$

Page 514, line 5: Should read $h_d \in L^2(A)$.

Page 514, lines 9–10:

$$\sum_{c\in D} \int_A f(c,a) h\big((c,a)^{-1}(d,a')\big)\,\mathrm{d}a$$
$$= \sum_{c\in D} \int_A f_c(a) h\big(c^{-1}d, \alpha(c, c^{-1}d)^{-1}[(d^{-1}c)\cdot a^{-1}]a'\big)\,\mathrm{d}a$$

Check. This boils down to showing

$$(c,a)^{-1}(d,a') = \big(c^{-1}d, \alpha(c, c^{-1}d)^{-1}[(d^{-1}c)\cdot a^{-1}]a'\big).$$

Recall that G has group product

$$(b,a)(c,a') = \big(bc, \alpha(b,c)\big(\gamma(c)^{-1}a\gamma(c)\big)a'\big)$$

and inverse

$$(b,a)^{-1} = \big(b^{-1}, [\gamma(b)a\gamma(b^{-1})]^{-1}\big).$$

So

$$\begin{aligned}
(c,a)^{-1}(d,a') &= \big(c^{-1}, [\gamma(c)a\gamma(c^{-1})]^{-1}\big)(d,a') \\
&= \big(c^{-1}d, \alpha(c^{-1},d)\big(\gamma(d)^{-1}[\gamma(c)a\gamma(c^{-1})]^{-1}\gamma(d)\big)a'\big) \\
&= \big(c^{-1}d, \alpha(c^{-1},d)\gamma(d)^{-1}\gamma(c^{-1})^{-1}a^{-1}\gamma(c)^{-1}\gamma(d)a'\big) \\
&= \big(c^{-1}d, \alpha(c^{-1},d)\big(\gamma(c^{-1})\gamma(d)\big)a^{-1}\gamma(c)^{-1}\gamma(d)a'\big) \\
&= \big(c^{-1}d, \alpha(c^{-1},d)\big(\gamma(c^{-1}d)\alpha(c^{-1},d)\big)a^{-1}\gamma(c)^{-1}\gamma(d)a'\big) \\
&= \big(c^{-1}d, \gamma(c^{-1}d)a^{-1}\gamma(c)^{-1}\gamma(d)a'\big)
\end{aligned}$$

Now observe that

$$\gamma(c)\gamma(c^{-1}d) = \gamma(cc^{-1}d)\alpha(c, c^{-1}d) = \gamma(d)\alpha(c, c^{-1}d),$$

and thus

$$\gamma(c)^{-1}\gamma(d)\alpha(c, c^{-1}d) = \gamma(c^{-1}d),$$

and hence

$$\gamma(c)^{-1}\gamma(d) = \gamma(c^{-1}d)\alpha(c, c^{-1}d)^{-1}.$$

Therefore

$$\begin{aligned}(c, a)^{-1}(d, a') &= \big(c^{-1}d, \gamma(c^{-1}d)^{-1}a^{-1}\gamma(c^{-1}d)\alpha(c, c^{-1}d)^{-1}a'\big)\\ &= \big(c^{-1}d, (d^{-1}c)\cdot a^{-1}\alpha(c, c^{-1}d)^{-1}a'\big)\\ &= \big(c^{-1}d, \alpha(c, c^{-1}d)^{-1}(d^{-1}c)\cdot a^{-1}a'\big),\end{aligned}$$

where we have used the fact that A is abelian. This proves the desired equality.

Remark 0.4. Note that we did not need to use the cocycle identity in the above. We also note that the remaining calculations in the integral on p. 514 follow from translation-invariance of the integral.

Page 514, lines 17–19 showing that $\Psi\lambda_f^G\Psi^{-1}$ is in the range of $\mathcal{M}$.

Check. We define a matrix M by $M_{d,c} = \widehat{\eta_{dc^{-1},d}}$, where η is defined by

$$\eta_{c,d}(a) = f_c\big((c^{-1}d)\cdot(\alpha(c, c^{-1}d)^{-1}a)\big) \quad \text{for each } a \in A.$$

Then

$$\big(\mathcal{M}(M)h\big)_d = \sum_{c\in D} M_{d,c}\mathcal{P}(h_c) = \sum_{c\in d} \widehat{\eta_{dc^{-1},d}}\mathcal{P}(h_c).$$

From Proposition 2 proof: Check: Let $g \in L^2(G)$, and let $h = \Psi g$. We use Proposition 1 from the paper, **but with slightly different notation.** For $f \in L^1(G)$ we define $\eta_{c,d} \in L^1(A)$ to be a function equal to $f_c\big((c^{-1}d)\cdot\big(\alpha(c, c^{-1}d)^{-1}a\big)\big)$ for almost every $a \in A$. We have

$$\big((\Psi\lambda_f^G\Psi^{-1})(h)\big)_d = \big((\Psi\lambda_f^G)(g)\big)_d = \big(\Psi\big(\lambda_f^G(g)\big)\big)_d = \big(\Psi(f * g)\big)_d.$$

By Proposition 1 in the paper the above is equal to

$$\begin{aligned}\sum_{c\in D}\widehat{\eta_{dc^{-1},d}}\mathcal{P}(g_c) &= \sum_{c\in D}\widehat{\eta_{dc^{-1},d}}\mathcal{P}(\Psi^{-1}h)_c\\ &= \sum_{c\in D}\widehat{\eta_{dc^{-1},d}}\mathcal{P}(\mathcal{P}^{-1}h_c)\\ &= \sum_{c\in D}\widehat{\eta_{dc^{-1},d}}h_c.\end{aligned}$$

Defining $M\in M_n\big(C_0(\hat{A})\big)$ by $M_{c,d}=\widehat{\eta_{dc^{-1},d}}$, this becomes

$$\sum_{c\in D}M_{d,c}h_c=\big(\mathcal{M}(M)h\big)_d.$$

Altogether we have

$$\Psi\lambda_f^G\Psi^{-1}h=\mathcal{M}(M)h \text{ for all } h\in\mathcal{H}=\bigoplus_{c\in D}L^2(\hat{A}),$$

ie $\Psi\lambda_f^G\Psi^{-1}=\mathcal{M}(M)$, or

$$M=\mathcal{M}^{-1}\big(\Psi\lambda_f^G\Psi^{-1}\big)=\mathcal{F}(f).$$

Therefore $\mathcal{F}(f)_{b,c}=M_{b,c}=\widehat{\eta_{cb^{-1},b}}$. In particular, $\mathcal{F}(f)_{b,e}=\widehat{\eta_{be^{-1},b}}=\widehat{\eta_{b,b}}$. For a.e. $a\in A$ we have that $\eta_{b,b}(a)=f_b\big((b^{-1}b)\cdot\big(\alpha(b,b^{-1}b)^{-1}a\big)\big)=f_b(a)$. Thus $\mathcal{F}(f)_{b,e}=\widehat{f_b}$, which is the formula in line 27 of p. 515.

Page 514, the formula in Proposition 2: Check. We know that $\big(\mathcal{F}(f)\big)_{b,c}=\widehat{\eta_{bc^{-1},b}}$. Using the translation-invariance of the integral we calculate

$$\begin{aligned}\widehat{\eta_{c,d}}(\chi) &= \int_A f_c\big((c^{-1}d)\cdot\alpha(c,c^{-1}d)^{-1}a\big)\chi(a)\,\mathrm{d}a\\ &= \int_A f_c\big((c^{-1}d)\cdot a\big)\chi\big(\alpha(c,c^{-1}d)a\big)\,\mathrm{d}a\\ &= \chi\big(\alpha(c,c^{-1}d)\big)\int_A f_c\big((c^{-1}d)\cdot a\big)\chi(a)\,\mathrm{d}a,\end{aligned}$$

since χ is a homomorphism. This becomes

$$\begin{aligned}
&\chi\big(\alpha(c, c^{-1}d)\big) \int_A f_c(a)\chi\big((c^{-1}d)^{-1} \cdot a\big)\, \mathrm{d}a \\
&= \chi\big(\alpha(c, c^{-1}d)\big) \int_A f_c(a)\big((c^{-1}d) \cdot \chi)\big)(a)\, \mathrm{d}a \\
&= \chi\big(\alpha(c, c^{-1}d)\big) \widehat{f_c}\big((c^{-1}d) \cdot \chi\big).
\end{aligned}$$

Page 515, line 11: Then, for any $y \in G$ and $f \in L^1(G)$, $\rho(y)\lambda_f^G \rho(y)^* = \lambda_f^G$. Thus

$$\begin{aligned}
\big(\mathcal{F}(f)\big)_{b,c}(\chi) &= \widehat{\eta_{bc^{-1},b}}(\chi) \\
&= \chi\big(\alpha(bc^{-1}, cb^{-1}b)\big) \widehat{f_{bc^{-1}}}\big(((bc^{-1})^{-1}b) \cdot \chi\big) \\
&= \chi\big(\alpha(bc^{-1}, c)\big) \widehat{f_{bc^{-1}}}(c \cdot \chi),
\end{aligned}$$

which is the formula in Proposition 2.

Check: This is easy. Recall that $\lambda_f^G \in \mathcal{B}\big(L^2(G)\big)$ is defined by $\lambda_f^G(g) = f * g$, for all $g \in L^2(G)$. We sometimes write $\lambda_f^G = \lambda^G(f)$; then $\lambda^G : L^1(G) \to \mathcal{B}\big(L^2(G)\big)$. The map ρ is the unitary operator on $L^2(G)$ given by $\rho(y)g(x) = g(xy)$ for all $x, y \in G$. We also have $\rho(y)^{-1} = \rho(y^{-1})$ for any $y \in G$, because ρ is a homomorphism. We now calculate:

$$\begin{aligned}
[\rho(y)\lambda_f^G \rho(y)^*]g(x) &= \big(\rho(y)\lambda_f^G\big) g(xy^{-1}) \\
&= \rho(y)\big(\lambda_f^G(g)\big)(xy^{-1}) \\
&= \rho(y)(f * g)(xy^{-1}) \\
&= (f * g)(xy^{-1}y) \\
&= (f * g)(x) \\
&= \lambda_f^G(g)(x).
\end{aligned}$$

So $\rho(y)\lambda_f^G \rho(y)^* g = \lambda_f^G g$ for all $g \in L^2(G)$, and hence $\rho(y)\lambda_f^G \rho(y)^* = \lambda_f^G$ for all $y \in G$, as desired.

Page 515, lines 13–14: for any $b \in D$ and $\chi \in \hat{A}$,

$$\big(U(d)h\big)_b(\chi) = \overline{d^{-1} \cdot \chi(\alpha(b, d))} h_{bd}(d^{-1} \cdot \chi),$$

where $h \in \mathcal{H} = \bigoplus_{d\in D} L^2(\hat{A})$.

Check: Recall that in the paper ρ is the right regular representation of G on $L^2(G)$, *ie*, $\rho(y)g(x) = g(xy)$ for $g \in L^2(G)$, $x, y \in G$. Recall that the map U is defined by

$$U(d) = \Psi\rho\big(\gamma(d)\big)\Psi^{-1} \in \mathcal{B}\Big(\bigoplus_{d\in D} L^2(\hat{A})\Big)$$

for each $d \in D$. We let $g = \Psi^{-1}h \in L^2(G)$. From what we have established previously we know that for $b \in D$ we have

$$\begin{aligned}
\big[\Psi[\rho\big(\gamma(d)\big)\Psi^{-1}h]\big]_b &= \mathcal{P}\big[\big(\rho\big(\gamma(d)\big)\Psi^{-1}h\big)_b\big] \\
&= \mathcal{P}\big[\big(\rho\big(\gamma(d)\big)g\big)_b\big]
\end{aligned}$$

Note that for $a \in A$ and $b \in D$ we have

$$\begin{aligned}
\big[\rho\big(\gamma(d)\big)g\big]_b(a) &= \rho\big(\gamma(d)\big)g\big(\gamma(b)a\big) \\
&= g\big(\gamma(b)a\gamma(d)\big) \\
&= g\big(\gamma(b)\gamma(d)\underbrace{\gamma(d)^{-1}a\gamma(d)}_{=d^{-1}\cdot a}\big) \\
&= g\big(\gamma(bd)\alpha(b,d)d^{-1}\cdot a\big).
\end{aligned}$$

Putting these together we have

$$\begin{aligned}
\big[\Psi\rho\big(\gamma(d)\big)\Psi^{-1}h\big]_b(\chi) &= \int_A \big[\rho\big(\gamma(d)\big)g\big]_b(a)\chi(a)\,\mathrm{d}a \\
&= \int_A g\big(\gamma(bd)\alpha(b,d)d^{-1}\cdot a\big)\chi(a)\,\mathrm{d}a \\
&= \int_A g_{bd}\big(\alpha(b,d)(d^{-1}\cdot a)\big)\chi(a)\,\mathrm{d}a \\
&= \int_A g_{bd}\big(\alpha(b,d)a\big)\chi(d\cdot a)\,\mathrm{d}a,
\end{aligned}$$

where the last inequality follows from the translation-invariance of the Haar integral. Recall that the action of D on A induces an action on $\hat{A}$ given by $(d\cdot\chi)(a) = \chi(d^{-1}\cdot a)$.

With this and further use of translation-invariance the above becomes

$$\begin{aligned}\int_A g_{bd}\big(\alpha(b,d)a\big)(d^{-1}\cdot\chi)(a)\,\mathrm{d}a &= \int_A g_{bd}(a)(d^{-1}\cdot\chi)\big(\alpha(b,d)^{-1}a\big)\,\mathrm{d}a\\ &= (d^{-1}\cdot\chi)\big(\alpha(b,d)^{-1}\big)\int_A g_{bd}(a)(d^{-1}\cdot\chi)(a)\,\mathrm{d}a\\ &= (d^{-1}\cdot\chi)\big(\alpha(b,d)^{-1}\big)\big(\mathcal{P}(g_{bd}\big)(d^{-1}\cdot\chi)\\ &= \overline{(d^{-1}\cdot\chi)\big(\alpha(b,d)\big)}h_{bd}(d^{-1}\cdot\chi).\end{aligned}$$

Putting these together completes the calculation.

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$$\big(U(d)^*h\big)_b(\chi) = \chi\big(\alpha(bd^{-1},d)\big)h_{bd^{-1}}(d\cdot\chi).$$

Check: The calculation is similar to the previous one, but more involved. Since each of the maps making up $U(d)$ is unitary we have

$$U(d)^* = U(d)^{-1} = \Psi\rho\big(\gamma(d)\big)^{-1}\Psi^{-1};$$

since $\rho : G \to \mathcal{B}(L^2(G))$ is a homomorphism, the above becomes

$$U(d)^* = U(d)^{-1} = \Psi\rho\big(\gamma(d)^{-1}\big)\Psi^{-1}.$$

As in the previous calculation we let $h \in \mathcal{H} = \bigoplus_{d\in D} L^2(\hat{A})$ and $g = \Psi^{-1}h \in L^2(G)$. We have

$$\begin{aligned}\big[\Psi[\rho\big(\gamma(d)^{-1}\big)\Psi^{-1}h]\big]_b &= \mathcal{P}\big[\big(\rho\big(\gamma(d)^{-1}\big)\Psi^{-1}h\big)_b\big]\\ &= \mathcal{P}\big[\big(\rho\big(\gamma(d)^{-1}\big)g\big)_b\big]\end{aligned}$$

Note that for $a \in A$ and $b \in D$ we have

$$\begin{aligned}\big[\rho\big(\gamma(d)^{-1}\big)g\big]_b(a) &= \rho\big(\gamma(d)^{-1}\big)g\big(\gamma(b)a\big)\\ &= g\big(\gamma(b)a\gamma(d)^{-1}\big)\\ &= g\big(\gamma(b)\gamma(d)^{-1}\underbrace{\gamma(d)a\gamma(d)^{-1}}_{d\cdot a}\big).\end{aligned}$$

If we recall our formula for $\gamma(d)^{-1}$ the above becomes

$$\begin{aligned} g\big(\gamma(b)\gamma(d^{-1})\alpha(d,d^{-1})^{-1}d\cdot a\big) &= g\big(\gamma(bd^{-1})\alpha(b,d^{-1})\alpha(d,d^{-1})^{-1}d\cdot a\big) \\ &= g_{bd^{-1}}\big(\alpha(b,d^{-1})\alpha(d,d^{-1})^{-1}d\cdot a\big). \end{aligned}$$

Putting these together and using the translation-invariance of the integral as before we have

$$\begin{aligned} \big[\Psi\rho\big(\gamma(d)^{-1}\big)\Psi^{-1}h\big]_b(\chi) &= \int_A \big[\rho\big(\gamma(d)^{-1}\big)g\big]_b(a)\chi(a)\,\mathrm{d}a \\ &= \int_A g_{bd^{-1}}\big(\alpha(b,d^{-1})\alpha(d,d^{-1})^{-1}d\cdot a\big)\chi(a)\,\mathrm{d}a \\ &= \int_A g_{bd^{-1}}\big(\alpha(b,d^{-1})\alpha(d,d^{-1})^{-1}a\big)\chi(d^{-1}\cdot a)\,\mathrm{d}a \\ &= \int_A g_{bd^{-1}}\big(\alpha(b,d^{-1})\alpha(d,d^{-1})^{-1}a\big)(d\cdot\chi)(a)\,\mathrm{d}a \\ &= \int_A g_{bd^{-1}}(a)(d\cdot\chi)\big(\alpha(d,d^{-1})\alpha(b,d^{-1})^{-1}a\big)\,\mathrm{d}a \\ &= (d\cdot\chi)\big(\alpha(d,d^{-1})\alpha(b,d^{-1})^{-1}\big)\int_A g_{bd^{-1}}(a)(d\cdot\chi)(a)\,\mathrm{d}a \\ &= (d\cdot\chi)\big(\alpha(d,d^{-1})\alpha(b,d^{-1})^{-1}\big)h_{bd^{-1}}(d\cdot\chi). \end{aligned}$$

Now

$$\begin{aligned} (d\cdot\chi)\big(\alpha(d,d^{-1})\alpha(b,d^{-1})^{-1}\big) &= \chi\big(d^{-1}\cdot[\alpha(d,d^{-1})\alpha(b,d^{-1})^{-1}]\big) \\ &= \chi\big(\gamma(d)^{-1}\alpha(d,d^{-1})\alpha(b,d^{-1})^{-1}\gamma(d)\big). \end{aligned}$$

It is easy to verify that

$$\boxed{\alpha(d,d^{-1}) = \gamma(d)\alpha(d^{-1},d)\gamma(d)^{-1},}$$

whence

$$\gamma(d)^{-1}\alpha(d,d^{-1}) = \alpha(d^{-1},d)\gamma(d)^{-1}.$$

Thus

$$\begin{aligned}
(d\cdot\chi)\big(\alpha(d,d^{-1})\alpha(b,d^{-1})^{-1}\big) &= \chi\big(\alpha(d^{-1},d)\gamma(d)^{-1}\alpha(b,d^{-1})^{-1}\gamma(d)\big)\\
&= \chi\big(\alpha(d^{-1},d)[\gamma(d)^{-1}\alpha(b,d^{-1})\gamma(d)]^{-1}\big)\\
&= \chi\big(\alpha(b,e_D)\alpha(d^{-1},d)[d^{-1}\cdot\alpha(b,d^{-1})]^{-1}\big)\\
&= \chi\big(\alpha(b,d^{-1}d)\alpha(d^{-1},d)[d^{-1}\cdot\alpha(b,d^{-1})]^{-1}\big)\\
&= \chi\big(\alpha(bd^{-1},d)\big),
\end{aligned}$$

where in the third equality we use the fact that $\alpha(b,e_D)=e_G$, and in the last equality we have used the cocycle identity. This completes the calculation.

Page 515, lines 15–16 For $F=(F_{b,c})_{b,c\in D}$ in $M_n\big(C_0(\hat{A})\big)$,

$$[U(d)\mathcal{M}(F)U(d)^*h]_b(\chi)=\sum_{c\in D}(d^{-1}\cdot\chi)\big(\alpha(b,d)^{-1}\alpha(c,d)\big)F_{bd,cd}(d^{-1}\cdot\chi)h_c(\chi).$$

Check: We simply apply the maps. We recall that $\mathcal{M}:M_n\big(C_0(\hat{A})\big)\to\mathcal{B}\big(\bigoplus_{d\in D}L^2(\hat{A})\big)$ is defined for each $F=(F_{b,c})_{b,c\in D}$ in $M_n\big(C_0(\hat{A})\big)$ by $\big(\mathcal{M}(F)h\big)_b=\sum_{c\in D}F_{b,c}h_c$, for each $h\in\bigoplus_{d\in D}L^2(\hat{A})$. For such h and F we have

$$\begin{aligned}
\big(\mathcal{M}(F)U(d)^*h\big)_b(\chi) &= \Big(\sum_{c\in D}F_{b,c}\big(U(d)^*h\big)_c\Big)(\chi)\\
&= \sum_{c\in D}F_{b,c}(\chi)\big(U(d)^*h\big)_c(\chi)\\
&= \sum_{c\in D}F_{b,c}(\chi)\chi\big(\alpha(cd^{-1},d)\big)h_{cd^{-1}}(d\cdot\chi).
\end{aligned}$$

Thus

$$\begin{aligned}
&[U(d)\mathcal{M}(F)U(d)^*h]_b(\chi) = \big[\big(U(d)\big)\big(\mathcal{M}(F)U(d)^*h\big)\big]_b(\chi)\\
&\qquad= \overline{(d^{-1}\cdot\chi)(\alpha(b,d))}\big(\mathcal{M}(F)U(d)^*h\big)_{bd}(d^{-1}\cdot\chi)\\
&\qquad= (d^{-1}\cdot\chi)(\alpha(b,d)^{-1})\sum_{c\in D}F_{bd,c}(d^{-1}\cdot\chi)(d^{-1}\cdot\chi)\big(\alpha(cd^{-1},d)\big)h_{cd^{-1}}\big(d\cdot(d^{-1}\cdot\chi)\big).
\end{aligned}$$

If we make the shift of indices $\{cd : c \in D\} = \{c : c \in D\}$, the above becomes

$$\begin{aligned}
&(d^{-1} \cdot \chi)(\alpha(b,d)^{-1}) \sum_{c \in D} F_{bd,cd}(d^{-1} \cdot \chi)(d^{-1} \cdot \chi)\big(\alpha(c,d)\big)h_c(\chi) \\
= \quad & \sum_{c \in D} (d^{-1} \cdot \chi)\big(\alpha(b,d)^{-1}\alpha(c,d)\big)F_{bd,cd}(d^{-1} \cdot \chi)h_c(\chi),
\end{aligned}$$

where we have used the fact that $d^{-1} \cdot \chi$ is a homomorphism.

Page 515, line 17 Thus conjugation by $U(d)$ leaves the range of $\mathcal{M}$ invariant

Check: For $h \in \bigoplus_{d \in D} L^2(\hat{A})$ we must show that we can express $[U(d)\mathcal{M}(F)U(d)^*h]_b$ as the bth entry of $\mathcal{M}(G)h$ for some $G \in M_n\big(C_0(\hat{A})\big)$. Comparing the forumlas

$$\big(\mathcal{M}(F)h\big)_b(\chi) = \sum_{c \in D} F_{b,c}(\chi)h_c(\chi)$$

and

$$[U(d)\mathcal{M}(F)U(d)^*h]_b(\chi) = \sum_{c \in D} (d^{-1} \cdot \chi)\big(\alpha(b,d)^{-1}\alpha(c,d)\big)F_{bd,cd}(d^{-1} \cdot \chi)h_c(\chi),$$

we see that defining

$$G_{b,c}(\chi) = (d^{-1} \cdot \chi)\big(\alpha(b,d)^{-1}\alpha(c,d)\big)F_{bd,cd}(d^{-1} \cdot \chi) \in C_0(\hat{A}) \tag{6}$$

then

$$[U(d)\mathcal{M}(F)U(d)^*h]_b(\chi) = [\mathcal{M}(G)h]_b(\chi)$$

for all $\chi \in \hat{A}$ and all h, whence

$$U(d)\mathcal{M}(F)U(d)^*h = \mathcal{M}(G)h$$

for all h, *ie*,

$$U(d)\mathcal{M}(F)U(d)^* = \mathcal{M}(G). \tag{7}$$

This shows that the range of $\mathcal{M}$ is indeed invariant under conjugation by $U(d)$.

Remark 0.5. We note that the above matrix G also gives us the formula in line 19 of the same page. Indeed, we define $\beta : D \to \mathrm{Aut}\Big(\big(M_n\big(C_0(\hat{A})\big)\big)\Big)$ by

$$\beta(d)(F) = \mathcal{M}^{-1}\big(U(d)\mathcal{M}(F)U(d)^*\big) \quad \text{for all} \quad F \in M_n\big(C_0(\hat{A})\big),$$

(we verify later that $\beta(d)$ is in fact an automorphism,) and formula (7) says that

$$\beta(d)(F) = G.$$

Thus the coefficients of $\beta(d)(F)$ are given by (6).

Page 515, lines 19–20 further easy computations show that β is a homomorphism of D into the automorphism group of $M_n\big(C_0(\hat{A})\big)$.

Check: We must check that $\beta(d)$ is injective and surjective for each $d \in D$. We must check that $\beta(d)$ is a ring homomorphism. We must also check that β is a group homomorphism. As before, we let $\mathcal{H} = \bigoplus_{d\in D} L^2(\hat{A})$. We utilize the fact that $\mathcal{M}$ is a C^*-isomorphism of $M_n\big(C_0(\hat{A})\big)$ into $\mathcal{B}(\mathcal{H})$. We show that $\beta(d)$ is an isometry: since $U(d)$ is a unitary operator we have

$$\begin{aligned}
\|\beta(d)\| &= \|\mathcal{M}^{-1}\big(U(d)\mathcal{M}(F)U(d)^*\big)\| \\
&= \|U(d)\mathcal{M}(F)U(d)^*\| \\
&= \|\mathcal{M}(F)U(d)^*\| \\
&= \sup\{\|\mathcal{M}(F)U(d)^*h\| : h \in \mathcal{H}, \|h\| = 1\} \\
&= \sup\{\|\mathcal{M}(F)f\| : f \in L^2(G), \|f\| = 1\} \\
&= \|\mathcal{M}(F)\|.
\end{aligned}$$

Thus $\beta(d)$ is an isometry for each $d \in D$. $\beta(d)$ is linear because $\mathcal{M}$ and $U(d)$ are. If $\beta(d)(F) = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix, then $\mathcal{M}(F) = 0$ and therefore $F = \mathbf{0}$, which proves that $\beta(d)$ is injective. To see that $\beta(d)$ is surjective, we note that for any $F \in M_n\big(C_0(\hat{A})\big)$ we have that $F = \beta(d)[\mathcal{M}^{-1}\big(U(d)^*\mathcal{M}(F)U(d)\big)]$. We verify that $\beta(d)$ is multiplicative as follows: for any $F, G \in M_n\big(C_0(\hat{A})\big)$ we have

$$\begin{aligned}
\beta(d)(FG) &= \mathcal{M}^{-1}\big(U(d)\mathcal{M}(FG)U(d)^*\big) \\
&= \mathcal{M}^{-1}\big(U(d)\mathcal{M}(F)\mathcal{M}(G)U(d)^*\big) \\
&= \mathcal{M}^{-1}\big(U(d)\mathcal{M}(F)U(d)^*U(d)\mathcal{M}(G)U(d)^*\big) \\
&= \beta(d)(F)\beta(d)(G),
\end{aligned}$$

where we have used that $\mathcal{M}$ is multiplicative. So $\beta(d)$ is a bijective ring homomorphism from $M_n\big(C_0(\hat{A})\big)$ onto itself; *ie* $\beta(d)$ is an automorphism of this ring; it is in fact a $*$-homomorphism, as one can easily verify.

To check that β is a group homomorphism is a little more involved: we first see how $U(bc)$ comes out: we note that since $\gamma(b)\gamma(c) = \gamma(bc)\alpha(b,c)$, we have $\gamma(bc) = \gamma(b)\gamma(c)\alpha(b,c)^{-1}$. Thus

$$\begin{aligned} U(bc) &= \Psi\rho\big(\gamma(bc)\big)\Psi^{-1} \\ &= \Psi\rho\big(\gamma(b)\gamma(c)\alpha(b,c)^{-1}\big)\Psi^{-1} \\ &= \Psi\rho\big(\gamma(b)\big)\rho\big(\gamma(c)\big)\rho\big(\alpha(c,b)^{-1}\big)\Psi^{-1} \\ &= \Psi\rho\big(\gamma(b)\big)\Psi^{-1}\Psi\rho\big(\gamma(c)\big)\Psi^{-1}\Psi\rho\big(\alpha(c,b)^{-1}\big)\Psi^{-1} \\ &= U(b)U(c)\Psi\rho\big(\alpha(c,b)^{-1}\big)\Psi^{-1}. \end{aligned}$$

Thus for each $F \in M_n\big(C_0(\hat{A})\big)$

$$\begin{aligned} &\beta(bc)(F) = \mathcal{M}^{-1}\big(U(bc)\mathcal{M}(F)U(bc)^*\big) \\ &= \mathcal{M}^{-1}\big[U(b)U(c)\Psi\rho\big(\alpha(c,b)^{-1}\big)\Psi^{-1}\mathcal{M}(F)\Psi\rho\big(\alpha(c,b)\big)\Psi^{-1}U(c)^*U(b)^*\big] \\ &= \mathcal{M}^{-1}\big[U(b)U(c)\mathcal{M}(F)U(c)^*U(b)^*\big] \\ &= \mathcal{M}^{-1}\big[U(b)\mathcal{M}\big(\mathcal{M}^{-1}\big(U(c)\mathcal{M}(F)U(c)^*\big)\big)U(b)^*\big] \\ &= \beta(b)[\beta(c)(F)] \end{aligned}$$

The above requires verifying a step in greater detail: we claim that for any $a \in A$,

$$\Psi\rho(a^{-1})\Psi^{-1}\mathcal{M}(F)\Psi\rho(a)\Psi^{-1} = \mathcal{M}(F).$$

We first observe that for $a \in A$, $\big(\rho(a)g\big)_c = \rho(a)g_c$, because for any $x \in A$,

$$\big(\rho(a)g\big)_c(x) = \big(\rho(a)g\big)\big(\gamma(c)x\big) = g\big(\gamma(c)xa\big) = g_c\big(xa\big) = \rho(a)g_c(x).$$

So in this case $\Psi\rho(a)\Psi^{-1} = \bigoplus \mathcal{P}\rho(a)\mathcal{P}^{-1}$, and for $h = (h_c)_{c\in D} \in \mathcal{H}$ it suffices to determine how the above map acts on the bth component h_b. We also know from

Fourier analysis that $\mathcal{P}[\rho(a)g_c](\chi) = \overline{\chi(a)}\mathcal{P}(g_c)(\chi)$, for all $\chi \in \hat{A}$. If we let $g = \Psi^{-1}h$, so that $\widehat{g_b} = h_b$, then by definition $\mathcal{P}\rho(a)\mathcal{P}^{-1}h_b = \mathcal{P}\big(\rho(a)g_b\big)$. Now

$$\begin{aligned}
[\mathcal{M}(F)\Psi\rho(a)\Psi^{-1}h]_b(\chi) &= \sum_{c\in D} F_{b,c}(\chi)[\Psi\rho(a)\Psi^{-1}h]_c(\chi) \\
&= \sum_{c\in D} F_{b,c}(\chi)\mathcal{P}\rho(a)\mathcal{P}^{-1}h_c(\chi) \\
&= \sum_{c\in D} F_{b,c}(\chi)\mathcal{P}\big(\rho(a)g_c\big) \\
&= \sum_{c\in D} F_{b,c}(\chi)\overline{\chi(a)}\mathcal{P}(g_c)(\chi).
\end{aligned}$$

We must now apply $\mathcal{P}\rho(a^{-1})\mathcal{P}^{-1}$ to this. We must calculate

$$\mathcal{P}[\rho(a^{-1})\mathcal{P}^{-1}F_{b,c}(\chi)\overline{\chi(a)}(\mathcal{P}g_c)(\chi)]$$

First we note that for each $\omega \in A$,

$$\mathcal{P}^{-1}\big(F_{b,c}(\chi)\overline{\chi(a)}(\mathcal{P}g_c)(\chi)\big)(\omega) = \int_{\hat{A}} F_{b,c}(\chi)\overline{\chi(a)}(\mathcal{P}g_c)(\chi)\overline{\chi(\omega)}\,\mathrm{d}\chi.$$

and hence

$$\begin{aligned}
&\rho(a^{-1})\big[\mathcal{P}^{-1}\big(F_{b,c}(\chi)\overline{\chi(a)}(\mathcal{P}g_c)(\chi)\big)\big](\omega) \\
&\quad= \mathcal{P}^{-1}\big(F_{b,c}(\chi)\overline{\chi(a)}(\mathcal{P}g_c)(\chi)\big)(\omega a^{-1}) \\
&\quad= \int_{\hat{A}} F_{b,c}(\chi)\overline{\chi(a)}(\mathcal{P}g_c)(\chi)\overline{\chi(\omega a^{-1})}\,\mathrm{d}\chi \\
&\quad= \int_{\hat{A}} F_{b,c}(\chi)\overline{\chi(a)}(\mathcal{P}g_c)(\chi)\overline{\chi(\omega)\chi(a^{-1})}\,\mathrm{d}\chi \\
&\quad= \int_{\hat{A}} F_{b,c}(\chi)(\mathcal{P}g_c)(\chi)\overline{\chi(\omega)}\,\mathrm{d}\chi \\
&\quad= \mathcal{P}^{-1}\big(F_{b,c}(\mathcal{P}g_c)\big)(\omega) \\
&\quad= \mathcal{P}^{-1}(F_{b,c}h_c)(\omega)
\end{aligned}$$

Therefore taking $\mathcal{P}$ of this gives

$$\mathcal{P}[\rho(a^{-1})\mathcal{P}^{-1}F_{b,c}(\chi)\overline{\chi(a)}(\mathcal{P}g_c)(\chi)] = F_{b,c}h_c.$$

We have shown that

$$\big(\mathcal{P}\rho(a^{-1})\mathcal{P}^{-1}\big)[\mathcal{M}(F)\Psi\rho(a)\Psi^{-1}h]_b = \sum_{c\in D} F_{b,c}h_c,$$

whence

$$[\Psi\rho(a^{-1})\Psi\mathcal{M}(F)\Psi\rho(a)\Psi^{-1}h]_b = \sum_{c\in D} F_{b,c}h_c = [\mathcal{M}(F)h]_b.$$

This establishes what we wanted to show.

Page 515, line 24 Then, for any $f \in L^1(G)$, $\rho\big(\gamma(d)\big)\lambda_f^G\rho\big(\gamma(d)\big)^* = \lambda_f^G$ implies that $\mathcal{F}(f) \in M_n\big(C_0(\hat{A})\big)^D$.

Check: Recall that $\mathcal{F}$ was defined from $L^1(G)$ to $M_n\big(C_0(\hat{A})\big)$ by

$$\mathcal{F}(f) = \mathcal{M}^{-1}(\Psi\lambda_f^G\Psi^{-1})$$

for $f \in L^1(G)$, and that it was shown in Proposition 1 that $\mathcal{F}$ extends by continuity to a C^*-isomorphism of $C^*(G)$ onto a C^*-subalgebra of $M_n\big(C_0(\hat{A})\big)$. For any $f \in L^1(G)$ we have

$$\mathcal{F}(f) \in M_n\big(C_0(\hat{A})\big)^D \iff \beta(d)\big(\mathcal{F}(f)\big) = \mathcal{F}(f) \ \text{ for all } d \in D.$$

By definition,

$$\begin{aligned}\beta(d)\big(\mathcal{F}(f)\big) &= \beta(d)\big(\mathcal{M}^{-1}(\Psi\lambda_f^G\Psi^{-1})\big)\\ &= \mathcal{M}^{-1}\big[U(d)\mathcal{M}\big(\mathcal{M}^{-1}\Psi\lambda_f^G\Psi^{-1}\big)U(d)^*\big]\\ &= \mathcal{M}^{-1}\big[U(d)\Psi\lambda_f^G\Psi^{-1}U(d)^*\big].\end{aligned}$$

Since $U(d) = \Psi\rho\big(\gamma(d)\big)\Psi^{-1}$, this becomes

$$\begin{aligned}&\mathcal{M}^{-1}\big[\Psi\rho\big(\gamma(d)\big)\Psi^{-1}\Psi\lambda_f^G\Psi^{-1}\big(\Psi\rho\big(\gamma(d)\big)\Psi^{-1}\big)^*\big]\\ &\quad= \mathcal{M}^{-1}\big[\Psi\rho\big(\gamma(d)\big)\lambda_f^G\rho\big(\gamma(d)\big)^*\big]\\ &\quad= \mathcal{M}^{-1}\big(\Psi\lambda_f^G\Psi^{-1}\big)\\ &\quad= \mathcal{F}(f),\end{aligned}$$

as desired.

Pages 515–516, $\mathcal{F}\big(L^1(G)\big)$ is dense in $M_n\big(C_0(\hat{A})\big)^D$.

Check. We examine the first statement, which says that for fixed $F \in M_n\big(C_0(\hat{A})\big)^D$ and $\varepsilon > 0$, we can choose $\delta > 0$ so that $F' \in M_n\big(C_0(\hat{A})\big)$ with $\|F'_{b,c} - F_{b,c}\|_\infty < \delta$ for all $b, c \in D$ implies $\|F' - F\| < \varepsilon$. Recall the definition of $\|F\|$ for $F \in M_n\big(C_0(\hat{A})\big)$:

$$\|F\| = \sup_{x \in \hat{A}} \big\{\|F(x)\|\big\}.$$

Here the norm on the matrix $\|F(x)\|$ can be any norm, since all norms are equivalent on a finite-dimensional vector space. So let us fix this norm to be the max norm, *ie*

$$\|F(x)\|_{\max} = \sup_{b,c \in D} \|[F(x)]_{b,c}\| = \sup_{b,c \in D} \|\big(F_{b,c}\big)(x)\|.$$

Thus

$$\begin{aligned}
\|F' - F\| &= \sup_{x \in \hat{A}} \big\{\|\big(F' - F\big)(x)\|\big\} \\
&= \sup_{x \in \hat{A}} \sup_{b,c \in D} \big\{\|\big(F' - F\big)_{b,c \in D}(x)\|\big\} \\
&= \sup_{b,c \in D} \sup_{x \in \hat{A}} \big\{\|\big(F' - F\big)_{b,c \in D}(x)\|\big\} \\
&= \sup_{b,c \in D} \big\{\|\big(F' - F\big)_{b,c \in D}\|_\infty\big\} \\
&= \big\|\|\big(F' - F\big)_{b,c}\|_\infty\big\|_{\max}.
\end{aligned}$$

If, instead of the max norm on the $n \times n$ complex matrices we had chosen some other norm, such that the operator norm, then by the equivalence of the norms the above would simply become an inequality with some constant factor. Thus the ε–δ statement above becomes clear.

Now we consider the next statement: for each $b \in D$, pick $f_b \in L^1(A)$ such that $\|\widehat{f_b} - F_{b,e}\| < \delta$. We can do this because, as indicated earlier in the paper, $\{\hat{f} : f \in L^1(A)\}$ is dense in $\big(C_0(\hat{A}), \|\cdot\|_\infty\big)$. Also, since $\big(\mathcal{F}(f)\big)_{b,e} = \widehat{f_b}$, this means

$$\|\big(\mathcal{F}(f) - F\big)_{b,e}\|_\infty = \|\big(\mathcal{F}(f)\big)_{b,e} - F_{b,e}\|_\infty < \delta \quad \text{for all } b \in D. \tag{8}$$

The next sentence says: Define $f \in L^1(G)$ by $f(b,a) = f_b(a)$, for all $b \in D$, $a \in A$. In the terminology we defined in these notes, this is simply to say take $f = \theta^{-1}\big((f_b)_{b\in D}\big)$.

Now we verify the formula in the Remark on p. 516: for each $F \in M_n\big(C_0(\hat{A})\big)^D$ we have

$$F_{b,c}(\chi) = (c\cdot\chi)\big(\alpha(b,c^{-1})^{-1}\alpha(c,c^{-1})\big)F_{bc^{-1},e}(c\cdot\chi), \quad \text{for } \chi \in \hat{A} \text{ and } b,c \in D. \tag{9}$$

This equations comes from the fact that, by definition of F being in the fixed point algebra, $\beta(d)F = F$ for all $d \in D$. Indeed, if we recall equation (6):

$$[\beta(d)(F)]_{b,c}(\chi) = (d^{-1}\cdot\chi)\big(\alpha(b,d)^{-1}\alpha(c,d)\big)F_{bd,cd}(d^{-1}\cdot\chi)$$

and consider $\beta(c^{-1})F(\chi) = F(\chi)$ component-wise we get equation (9). **Remark:** formula (9) proves the statement that precedes it in the paper, and is also useful for proving the density argument, to which we now return.

From equation (9) we see that

$$\begin{aligned}
\|\big(\mathcal{F}(f)-F\big)_{b,c}\|_\infty &= \sup_{\chi\in\hat{A}} \|\big(\mathcal{F}(f)-F\big)_{b,c}(\chi)\| \\
&= \sup_{\chi\in\hat{A}} \Big\|(c\cdot\chi)\big(\alpha(b,c^{-1})^{-1}\alpha(c,c^{-1})\big)\big(\mathcal{F}(f)-F\big)_{bc^{-1},e}(c\cdot\chi)\Big\| \\
&= \sup_{\chi\in\hat{A}} \Big\|\big(\mathcal{F}(f)-F\big)_{bc^{-1},e}(c\cdot\chi)\Big\| \\
&= \Big\|\big(\mathcal{F}(f)-F\big)_{bc^{-1},e}\Big\|_\infty
\end{aligned}$$

Since we can make this less than δ for all $b,c \in D$, we have $\|\mathcal{F}f - F\| < \varepsilon$.

Page 516: The Example

Before addressing the example in Keith's paper, we give some background on the group pg which is used for that example. We let $\mathrm{GL}_n(\mathbb{R})$ denote the group of invertible linear transformations of $\mathbb{R}^n$ with identity element id. For $x \in \mathbb{R}^n$ and $L \in \mathrm{GL}_n(\mathbb{R})$, define $[x,L]$ on $\mathbb{R}^n$ by $[x,L]z = L(z+x)$. We define

$$\mathrm{Aff}(\mathbb{R}^n) = \{[x,L] : x \in \mathbb{R}^n, L \in \mathrm{GL}_n\mathbb{R}\};$$

this is a group with multiplication,

$$[x, L][y, M] = [M^{-1}x + y, LM]$$

inverse,

$$[x, L]^{-1} = [-Lx, L^{-1}]$$

and identity $[0, \mathrm{id}]$, the identity transformation on $\mathbb{R}^n$. The group of translations in $\mathrm{Aff}(\mathbb{R}^n)$, explicitly

$$\{[x, \mathrm{id}] : x \in \mathbb{R}^n\},$$

is a normal subgroup which we denote by $\mathrm{Trans}(\mathbb{R}^n)$. Let $q : \mathrm{Aff}(\mathbb{R}^n) \to \mathrm{GL}_n(\mathbb{R})$ be given by $q([x, L]) = L$; q is a homomorphism onto $\mathrm{GL}_n(\mathbb{R})$ with kernel $\mathrm{Trans}(\mathbb{R}^n)$, so that

$$\mathrm{Aff}(\mathbb{R}^n) \Big/ \mathrm{Trans}(\mathbb{R}^n) \cong \mathrm{GL}_n(\mathbb{R}).$$

Let $\mathcal{O}_n$ denote the group of orthogonal transformations of $\mathbb{R}^n$ and let

$$\mathrm{Isom}(\mathbb{R}^n) = \{[x, L] \in \mathrm{Aff}(\mathbb{R}^n) : L \in \mathcal{O}_n\};$$

this is a closed subgroup of $\mathrm{Aff}_n(\mathbb{R}^n)$. A subgroup Γ of $\mathrm{Isom}(\mathbb{R}^n)$ is called a *crystal group* if it is discrete and co-compact, *ie* $\mathbb{R}^n/\Gamma$ is compact, where, by definition,

$$\mathbb{R}^n/\Gamma = \{[z] : z \in \mathbb{R}^n\},$$

where $[z] = \Gamma z = \big\{[x, L]z : [x, L] \in \Gamma\big\}$, the Γ-orbit of z in $\mathrm{Aff}_n(\mathbb{R})$. We make the collection of orbits into a group by defining the group product on representatives of equivalence classes.

Exercise 0.6. Γ is co-compact if and only if there exists a compact $K \subset \mathbb{R}^n$ such that $\bigcup_{[x,L]\in\Gamma} [x, L]K = \mathbb{R}^n$ (and this union is *ae*-disjoint—check!)

We give $\mathbb{R}^n/\Gamma$ the quotient topology, *ie*, the strongest topology that makes the canonical quotient map $z \to \Gamma z \in \mathbb{R}^n/\Gamma$ continuous.

Let Γ be a crystal group, and let

$$D = q(\Gamma) = \{L \in \mathcal{O}_n : [x, L] \in \Gamma \text{ for some } x \in \mathbb{R}^n\}.$$

Then D is a finite subgroup of $\mathcal{O}_n$ called the *point group.* We define

$$A = \Gamma \cap \mathrm{Trans}(\mathbb{R}^n) = \{[x, L] \in \Gamma : L = \mathrm{id}\};$$

this is an abelian normal subgroup of Γ, and is the kernel of $q|_\Gamma$; thus $\Gamma\big/A \cong D$, by the first isomorphism theorem.

Exercise 0.7. There exists $\{u_1, u_2, \ldots, u_n\} \subset \mathbb{R}^n$ such that $\mathrm{span}\{u_1, u_2, \ldots, u_n\} = \mathbb{R}^n$, and

$$A = \Big\{\Big[\sum_{j=1}^{n} k_j u_j, \mathrm{id}\Big] : (k_1, \ldots, k_n) \in \mathbb{Z}^n\Big\},$$

so that $A \cong \mathbb{Z}^n$.

The example described in the paper is the two-dimenional crystal group $\Gamma = pg$, illustrated below. If we choose u_1 as a basis vector parallel to the horizontal axis and u_2 as a basis vector parallel to the vertical axis, then this group tiles the plane by integer shifts and a half-integer *glide reflection.* A glide reflection is what it sounds like: a reflection together with a translation (or glide). Explicitly, the glide reflection is $[(k+\frac{1}{2})u_1, \sigma]$, where σ is the element of $\mathcal{O}_n$ that fixes u_1 and sends u_2 to $-u_2$. We know from algebra that Γ is partitioned by the cosets of A. The cosets are fibers over elements of D, with respect to the homomorphism $q|_\Gamma$. For each $L \in D$, we let $\gamma(L)$ be a coset representative for the coset that is the fiber over L. Thus

$$\Gamma = \bigcup_{L \in D} \gamma(L)A.$$

Then $\gamma : D \to \Gamma$ is such that $q \circ \gamma = \mathrm{id}_D$. γ is a *cross-section* for the equivalence relation $\sim$ on Γ given by $[x, M] \sim [y, N] \iff q([x, M]) = q([y, N]) \iff L = M$; $\gamma(D)$ is a *transversal* for $\sim$ —a subset of Γ that meets each coset of A exactly once. This is the same meaning γ has had throughout.

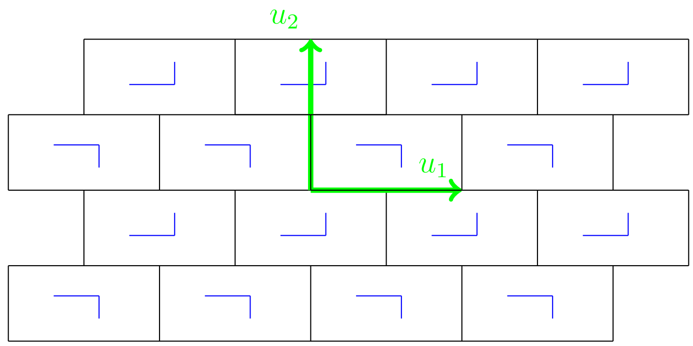

Figure 1: The wallpaper group pg. The vectors u_1 and u_2 form a basis for $\mathbb{R}^2$.

For the group $\Gamma = pg$, we can take $\gamma(\mathrm{id}_{\mathcal{O}_n}) = [0, \mathrm{id}_{\mathcal{O}_n}]$ and $\gamma(\sigma) = [\frac{1}{2}u_1, \sigma]$. We have

$$pg = \gamma(\mathrm{id})A \cup \gamma(\sigma)A,$$

or

$$pg = \{[ku_1 + ju_2, \mathrm{id}] : k, j \in \mathbb{Z}\} \cup \{[(k + \tfrac{1}{2})u_1 + ju_2, \sigma] : k, j \in \mathbb{Z}\}.$$

We have also seen in the paper by Keith Taylor how the group D acts on A: for $d \in D$ and $a \in A$, we have $d \cdot a = \gamma(d)a\gamma(d)^{-1}$; and how this gives rise to an action of D on $\hat{A}$: $d \cdot \chi = \chi(d \cdot a)$ for each $\chi \in \hat{A}$. We now introduce the notation of the example in the paper: we note that the group $\Gamma = pg$ we described above can be expressed as

$$G = \{(1, m, n), (-1, m, n) : m, n \in \mathbb{Z}\},$$

with group product given by the following table (note multiplication is row · column—this group is noncommutative):

row header · column header	$(1, m, n)$	$(-1, m, n)$
$(1, k, l)$	$(1, k + m, l + n)$	$(-1, k + m, n - l)$
$(-1, k, l)$	$(-1, k + m, l + n)$	$(1, k + m, n - l)$

This description can be realized under the mappings

$$[ku_1 + ju_2, \mathrm{id}] \mapsto (1, j, k),$$
$$[(k + \tfrac{1}{2})u_1 + ju_2, \sigma] \mapsto (-1, j, k).$$

We have $A = \{(1, m, n) : m, n \in \mathbb{Z}\} \cong \mathbb{Z}^2$, and $D = \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$. Identifying A with $\mathbb{Z}^2$ via $(1, m, n) \mapsto (m, n)$, the dual group $\hat{A} \cong \{\chi^\omega : (m, n) \mapsto \mathrm{e}^{2\pi\imath\omega\cdot(m,n)} : m, n \in \mathbb{Z}\}$ consists of homomorphisms from $\mathbb{Z}^2$ into S^1 that are $\mathbb{Z}^2$-periodic in the ω domain, and hence

$$\hat{A} \cong \left\{\chi^\omega : \chi^\omega(m, n) = \mathrm{e}^{2\pi\imath\omega\cdot(m,n)} \;\text{ and }\; \omega \in [-\tfrac{1}{2}, \tfrac{1}{2})^2\right\} \cong \mathbb{T}^2,$$

where $\mathbb{T}^2$ is the torus:

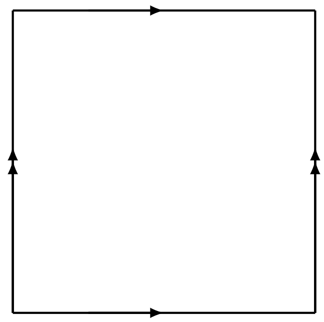

Figure 2: The Torus

The torus can be parametrized over the domain $[-\frac{1}{2}, \frac{1}{2})^2 \subset \mathbb{R}^2$ as above or, alternatively, it can be expressed in terms of complex parameters:

$$\mathbb{T}^2 = \{(z, w) : z, w \in \mathbb{C}, |z| = |w| = 1\} = S^1 \times S^1.$$

The latter parametrization is can be deduced from the former by writing $\omega = (\omega_1, \omega_2)$ so that

$$\mathrm{e}^{2\pi\imath\omega\cdot(m,n)} = \mathrm{e}^{2\pi\imath\omega_1 m}\mathrm{e}^{2\pi\imath\omega_2 n} = z^m w^n,$$

where $z = \mathrm{e}^{2\pi\imath\omega_1}$ and $w = \mathrm{e}^{2\pi\imath\omega_1}$. Thus for $(z, w) \in \mathbb{T}^2$ we have $\chi^{z,w} \in \hat{A}$ defined by $\chi^{z,w}(1, m, n) = z^m w^n$ for all $(1, m, n) \in A$.

A remark on the representation of $C^*(G)$ as the algebra of matrix-valued functions of the form given in Keith's paper: it would be better notation to express the matrix entries in terms of F_{k1} where $k = \pm 1$, since it was remarked earlier that the algebra can be realized as the matrices $\{F_{cd}\}$ whose *e-column* $\{F_{ce} : c \in D\}$ determines the whole matrix by D-invariance, according to the formula (9). So it would be more consistent notation to say that $C^*(G)$ is isomorphic to the algebra of matrix-valued functions F on $\mathbb{T}^2$ of the form:

$$F(z, w) = \begin{pmatrix} F_{11}(z, w) & zF_{-11}(z, \overline{w}) \\ F_{-11}(z, w) & F_{11}(z, \overline{w}) \end{pmatrix}. \tag{10}$$

Then the map which restricts F in $M_2\big(C(\mathbb{T}^2)\big) = C\big(\mathbb{T}^2, M_2(\mathbb{C})\big)$ to Ω pro-

vides an isomorphism of $C^*(G)$ with

$$\{F \in C\big(\Omega, M_2(\mathbb{C})\big) : F(z, \pm 1) \in \mathcal{R}_z, \quad \textbf{for each } z \in \mathbb{T}\},$$

where

$$\Omega = \{(z, w) \in \mathbb{T}^2 : \mathbf{Im}(w) \geq 0\},$$

and

$$\mathcal{R}_z = \left\{ \begin{pmatrix} a & zb \\ b & a \end{pmatrix} : a, b \in \mathbb{C} \right\},$$

a *-subalgebra of $M_2(\mathbb{C})$.

Check: We must verify that the map is one-to-one, onto, and an algebra homomorphism. We know that $C^*(G)$ is isomorphic to the algebra of matrix-valued functions F on $\mathbb{T}^2$ of the form (10) above. Let's first show that $F \mapsto F|_\Omega$ is onto. Let

$$\mathscr{G} = \{F \in C\big(\Omega, M_2(\mathbb{C})\big) : F(z, \pm 1) \in \mathcal{R}_z, \quad \text{for each } z \in \mathbb{T}\},$$

and let

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \in \mathscr{G}.$$

Let $\Omega^* = \{(z, \overline{w}) : (z, w) \in \Omega\}$, so that $\mathbb{T}^2 = \Omega \cup \Omega^*$, and this union overlaps only when $(z, w) = (z, \overline{w})$, which is to say, on $\{(z, \pm 1) : z \in S^1\}$. Consider the element of $H \in M_2\big(C(\mathbb{T}^2)\big)$ defined by

$$H(z, w) = \begin{cases} \begin{pmatrix} F_{22}(z, \overline{w}) & zF_{21}(z, \overline{w}) \\ \overline{z}F_{12}(z, \overline{w}) & F_{11}(z, \overline{w}) \end{pmatrix} & \text{if } (z, w) \in \Omega^* \\ \begin{pmatrix} F_{11}(z, w) & F_{12}(z, w) \\ F_{21}(z, w) & F_{22}(z, w) \end{pmatrix} & \text{if } (z, w) \in \Omega\,. \end{cases}$$

This is our candidate for an element in the preimage. Note that H is well-defined because $F \in \mathscr{G}$, which implies that $F_{22}(z, \pm 1) = F_{11}(z, \pm 1)$, and $F_{12}(z, \pm 1) = zF_{21}(z, \pm 1)$.

For $(z,w) \in \Omega^*$ we have $(z,\overline{w}) \in \Omega$, and hence

$$H_{22}(z,w) = F_{11}(z,\overline{w}) = H_{11}(z,\overline{w}) \quad \text{for } (z,w) \in \Omega^*;$$

and moreover, for $(z,w) \in \Omega$ we have $(z,\overline{w}) \in \Omega^*$, so that

$$H_{22}(z,w) = F_{22}(z,w) = F_{22}(z,\overline{\overline{w}}) = H_{11}(z,\overline{w}) \quad \text{for } (z,w) \in \Omega.$$

Thus $H_{22}(z,w) = H_{11}(z,\overline{w})$ for all $(z,w) \in \mathbb{T}^2$. Similarly,

$$H_{12}(z,w) = zF_{21}(z,\overline{w}) = zH_{21}(z,\overline{w}) \quad \text{for } (z,w) \in \Omega^*;$$

and

$$H_{12}(z,w) = F_{12}(z,w) = z[\overline{z}F_{12}(z,\overline{\overline{w}})] = zH_{21}(z,\overline{w}) \quad \text{for } (z,w) \in \Omega,$$

whence $H_{12}(z,w) = zH_{21}(z,\overline{w})$ for all $(z,w) \in \mathbb{T}^2$. Therefore H corresponds to an element of $C^*(G)$. Finally, that $H|_\Omega = F$ is immediate from the way we defined H. Thus we have shown that $F \mapsto F|_\Omega$ is onto.

Next we note that $F \mapsto F|_\Omega$ is trivially seen to be an algebra homomorphism. From this it follows that to verify that it is one-to-one, we need only check that it has trivial kernel. Suppose that $F|_\Omega = \mathbf{0}$. the zero matrix in $M_2\big(C(\mathbb{T}^2)\big)$, where

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \in M_2\big(C(\mathbb{T}^2)\big)\,.$$

Since F corresponds to an element in $C^*(G)$, we know from (10) that

$$F_{22}(z,w) = F_{11}(z,\overline{w}) \quad \text{for all } (z,w) \in \mathbb{T}^2, \tag{11}$$

and that

$$F_{12}(z,w) = zF_{21}(z,\overline{w}) \quad \text{for all } (z,w) \in \mathbb{T}^2. \tag{12}$$

Thus

$$F(z,w) = \begin{pmatrix} F_{11}(z,w) & zF_{21}(z,\overline{w}) \\ F_{21}(z,w) & F_{11}(z,\overline{w}) \end{pmatrix} \quad \text{for all } (z,w) \in \mathbb{T}^2,$$

and this is equal to the zero matrix when $(z,w) \in \Omega$. This implies that

$$F_{11}(z,w) = F_{11}(z,\overline{w}) = 0 \quad \text{when } (z,w) \in \Omega,$$

which is to say that $F_{11} = 0$ on $\Omega \cup \Omega^* = \mathbb{T}^2$. It follows from (11) that $F_{22} = 0$ on all of $\mathbb{T}^2$. Moreover,

$$zF_{21}(z,\overline{w}) = F_{21}(z,w) = 0 \quad \text{when } (z,w) \in \Omega,$$

whence $F_{21} = 0$ on all of $\mathbb{T}^2$; it follows from (12) that $F_{12} = 0$ on $\mathbb{T}^2$ as well. We've shown that $F = \mathbf{0}$, which concludes the argument.

From The Addendum

Page 517, line 7 As noted before, we will use the letter θ instead of U, since U has been used elsewhere. Thus for $h \in L^2(G)$ we have

$$\theta h = \big((\theta h)_d\big)_{d \in D},$$

where

$$(\theta h)_d(t) = h_d(t) = h(d,t).$$

Page 517, second paragraph The maps ρ and λ^G were defined on pages 515 and 512 of the paper, respectively.

Page 517, third paragraph Notice the similarity between the map $\tilde{\lambda}$ defined here and the map $\mathcal{M}$ defined earlier in the paper.

Page 517, line 20 It is easy to see that for $f \in L^1(G)$, $U\lambda^G(f)U^* \in M_n\big(C^*_\lambda(N)\big)$.

Check: As mentioned above, we use θ instead of U. We must show that

$$\theta\lambda^G(f)\theta^* \in \text{range}(\tilde{\lambda}) \quad \text{for all } f \in L^1(G).$$

So we must find a matrix $F = (F_{b,c})_{b,c \in D}$ such that

$$\theta\lambda^G(f)\theta^* = \tilde{\lambda}\big((F)_{b,c}\big).$$

This can be expressed componentwise as

$$\big(\theta\lambda^G(f)\theta^{-1}(h_b)_{b\in D}\big)_d = \big(\tilde{\lambda}(F)(h_b)_{b\in D}\big)_d = \sum_{b\in D}\lambda^N(F_{d,b})h_b = \sum_{b\in D}F_{d,b} * h_b,$$

for each $(h_c)_{c\in D} \in \bigoplus_{c\in D} L^2(N)$, and where the convolution above is over the group N.

First we note from (4) that

$$\theta^{-1}\big((h_b)_{b\in D}\big)_d = \sum_{b\in D}\widetilde{h_b} \in L^2(G),$$

where

$$\widetilde{h_b}(t) = h(t)\mathbb{1}_{\gamma(b)N}(t)$$

for all $t \in G$. Now we have

$$\lambda^G(f)\theta^{-1}(h_b)_{b\in D} = \lambda^G(f)\big(\sum_{b\in D}\widetilde{h_b}\big) = f * \big(\sum_{b\in D}\widetilde{h_b}\big) = \sum_{b\in D} f * \widetilde{h_b},$$

where the convolution here is over G. Writing $f = \sum_{c\in D}\widetilde{f_c}$, we have

$$\lambda^G(f)\theta^{-1}(h_b)_{b\in D} = \sum_{b\in D}\big(\sum_{c\in D}\widetilde{f_c}\big) * \widetilde{h_b}.$$

We wish to find the d-th component of this, *ie*

$$\big(\theta\lambda^G(f)\theta^{-1}(h_b)_{b\in D}\big)_d = \big[\sum_{b\in D}\big(\sum_{c\in D}\widetilde{f_c}\big) * \widetilde{h_b}\big]_d.$$

For any $t \in N$, we have

$$\big[\sum_{b\in D}\big(\sum_{c\in D}\widetilde{f_c}\big) * \widetilde{h_b}\big]_d(t) = \sum_{b\in D}\big[\big(\sum_{c\in D}\widetilde{f_c}\big) * \widetilde{h_b}\big]\big(\gamma(d)t\big).$$

Thus we would be done if we could write

$$\big[\big(\sum_{c\in D}\widetilde{f_c}\big) * \widetilde{h_b}\big]\big(\gamma(d)t\big) = (F_{d,b} * h_b)(t)$$

for any $t \in N$, for an appropriate $F_{d,b} \in L^1(N)$. The left-hand side is equal to

$$\begin{aligned}
&\int_G \big(\sum_{c\in D} \widetilde{f}_c\big)(y)\widetilde{h}_b\big(y^{-1}\gamma(d)t\big)\,\mathrm{d}y \\
&= \int_G \big(\sum_{c\in D} f(y)\mathbb{1}_{\gamma(c)N}(y)\big)h\big(y^{-1}\gamma(d)t\big)\mathbb{1}_{\gamma(b)N}\big(y^{-1}\gamma(d)t\big)\,\mathrm{d}y \\
&= \int_G \big[\sum_{c\in D} f\big(\gamma(d)y\big)\mathbb{1}_{\gamma(c)N}\big(\gamma(d)y\big)\big]h(y^{-1}t)\mathbb{1}_{\gamma(b)N}(y^{-1}t)\,\mathrm{d}y \\
&= \int_G \big[\sum_{c\in D} f\big(\gamma(d)y\big)\mathbb{1}_{\gamma(c)N}\big(\gamma(d)y\big)\big]h(y^{-1}t)\mathbb{1}_{N\gamma(b)^{-1}}(y)\,\mathrm{d}y \\
&= \int_G \big[\sum_{c\in D} f\big(\gamma(d)y\gamma(b)^{-1}\big)\mathbb{1}_{\gamma(c)N}\big(\gamma(d)y\gamma(b)^{-1}\big)\big]h\big(\gamma(b)y^{-1}t\big)\mathbb{1}_N(y)\,\mathrm{d}y \\
&= \int_N \big[\sum_{c\in D} f\big(\gamma(d)y\gamma(b)^{-1}\big)\mathbb{1}_{\gamma(c)N}\big(\gamma(d)y\gamma(b)^{-1}\big)\big]h_b\big(y^{-1}t\big)\,\mathrm{d}y \\
&= \big(F_{d,b} * h_b\big)(t),
\end{aligned}$$

where

$$F_{d,b}(y) = \sum_{c\in D} f\big(\gamma(d)y\gamma(b)^{-1}\big)\mathbb{1}_{\gamma(c)N}\big(\gamma(d)y\gamma(b)^{-1}\big) \in L^1(N);$$

in the above string of equalities we have used the left and right translation invariance of the Haar integral on the (unimodular) group G. The formula for $F_{d,b}$ may be simplified as follows:

$$\mathbb{1}_{\gamma(c)N}\big(\gamma(d)y\gamma(b)^{-1}\big) = \mathbb{1}_{\gamma(d)^{-1}\gamma(c)N\gamma(b)}(y),$$

and we are integrating over N, so the above indicator function is nonzero only when

$$y \in N \cap \gamma(d)^{-1}\gamma(c)N\gamma(b).$$

This holds precisely when

$$\gamma(d)^{-1}\gamma(c)\gamma(b) \in N.$$

If $c = db^{-1}$ then one can check that

$$\gamma(d)^{-1}\gamma(c)\gamma(b) = \gamma(b)^{-1}\alpha(b, b^{-1})\alpha(d, b^{-1})^{-1}\gamma(b) \in N.$$

Since the cosets partition G, this is the only value of c for which $\mathbb{1}_{\gamma(c)N}\big(\gamma(d)y\gamma(b)^{-1}\big)$ is nonzero. Thus we have

$$F_{d,b}(y) = f\big(\gamma(d)y\gamma(b)^{-1}\big)\,\mathbb{1}_{\gamma(db^{-1})N}\big(\gamma(d)y\gamma(b)^{-1}\big).$$