Wavelets in $\mathbb{R}^d$

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1 Wavelets and Multi-Resolution Analyses

These notes are based closely on Chapter 5 of Wojtaszczyk's book [8].

Definition 1.1. A fixed integer matrix A will be called a *dilation matrix* if all of its eigenvalues have absolute value or modulus greater than 1. For convenience we define the dilation operator on $L^2(\mathbb{R}^d)$ by $D_A f(x) = |\det(A)|^{1/2} f(Ax)$; this is a unitary operator on $L^2(\mathbb{R}^d)$.

Definition 1.2. Given a dilation matrix A, we say that a collection of functions $\{\Psi^1, \dots, \Psi^s\}$ with $\Psi^r \in L^2(\mathbb{R}^d)$ for $r = 1, \dots, s$ is a *multiwavelet* or, in some texts, a *wavelet set* associated to a A, if

$$\left\{ |\det(A)|^{j/2} \Psi^r(A^j x - \gamma) \right\} \tag{1}$$

for $r = 1, \dots, s$, $j \in \mathbb{Z}$, and $\gamma \in \mathbb{Z}^d$ is an orthonormal basis for $L^2(\mathbb{R}^d)$; in this case we call (1) a *wavelet system* associated to A. In terms of previously defined notation, we say that $\{\Psi^1, \dots, \Psi^s\}$ is a multiwavelet if the wavelet system

$$\left\{ D_{A^j}(T_\gamma \Psi^r) : r = 1, \dots, s, j \in \mathbb{Z}, \gamma \in \mathbb{Z}^d \right\}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$.

Definition 1.3. We say that a collection $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ is a *generalized multiresolution analysis* (GRMA) if it satisfies

(i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots$

(ii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(iv) $V_j = D_A^j(V_0) = D_{A^j}(V_0)$

(v) $f \in V_0$ implies $T_\gamma f \in V_0$ for all $\gamma \in \mathbb{Z}^d$.

If, in addition, we have

(vi) there exists a function $\Phi \in V_0$ such that $\{\Phi_\gamma\}_{\gamma\in\mathbb{Z}^d}$ is an orthonormal basis for V_0,

then the collection $\{V_j\}_{j\in\mathbb{Z}}$ is called a *multiresolution analysis* (MRA) associated to A, and Φ is the associated *scaling function*.

In this section we will show how to construct a multiwavelet from a given MRA. I include this because I think it's impressive as well as instructive. There are more abstract proofs of the existence of a multiwavelet given a GMRA and vice-versa (see [1] and [2]), but I think it's worthwhile to see more explicitly how to construct one. We will need some more terminology and some preliminary results first.

Proposition 1.4. Let A be a dilation matrix. The number of cosets of $A(\mathbb{Z}^d)$ in $\mathbb{Z}^d$ is $|\det(A)|$.

This Proposition is crucial. We follow closely the proof given in Mark Pinsky's book [6], pp. 355–356. We first need a lemma (also in Pinsky):

Lemma 1.5. Let $Q \subset \mathbb{R}^d$ be a measurable subset such that the $\mathbb{Z}^d$-translates of Q cover $\mathbb{R}^d$, i.e.,

$$\bigcup_{\gamma\in\mathbb{Z}^d} Q + \gamma = \mathbb{R}^d.$$

Then $m(Q) \geq 1$, with equality if and only if

$$m\big(Q \cap (Q+\gamma)\big) = 0 \quad \text{whenever } \gamma \neq 0.$$

Proof. Letting $\chi_Q(x)$ denote the indicator function on Q, and define

$$f(x) := \sum_{\gamma \in \mathbb{Z}^d} \chi_Q(x+\gamma).$$

We have that

$$\begin{aligned}
m(Q) &= \int_{\mathbb{R}^d} \chi_Q(x)\,\mathrm{d}m(x) \\
&= \int_{\mathbb{R}^d} \sum_{\gamma \in \mathbb{Z}^d} \chi_{[0,1]^d+\gamma}(x)\chi_Q(x)\,\mathrm{d}m(x) \\
&= \int_{\mathbb{R}^d} \chi_{[0,1]^d}(x) \sum_{\gamma \in \mathbb{Z}^d} \chi_Q(x+\gamma)\,\mathrm{d}m(x) \\
&= \int_{[0,1]^d} f(x)\,\mathrm{d}m(x) \\
&\geq 1, \qquad (2)
\end{aligned}$$

since $f(x) \geq 1$ everywhere. If $m(Q) = 1$, then the above equations implies that

$$\int_{[0,1]^d} \big(f(x) - 1\big)\,\mathrm{d}m(x) = 0,$$

and hence $f(x) = 1$ a.e. on $[0,1]^d$; since f is $\mathbb{Z}^d$-periodic, this means that $f = 1$ a.e. in $\mathbb{R}^d$. If $m\big(Q \cap (Q+\gamma)\big) > 0$, then

$$\begin{aligned}
f(x) &\geq \chi_Q(x) + \chi_Q(x-\gamma) \\
&= \chi_Q(x) + \chi_{Q+\gamma}(x) \\
&= 2 \quad \text{for } x \in Q \cap (Q+\gamma),
\end{aligned}$$

a contradiction.

Suppose conversely that $m\big(Q \cap (Q+\gamma)\big) = 0$ for each $\gamma \neq 0$. This implies that the function f defined above is less than or equal to 1 a.e.. For suppose that $f \geq 2$ on some set E of positive measure. Then since $\bigcup_{\gamma \in \mathbb{Z}^d} Q + \gamma = \mathbb{R}^d$, we have

$$E = E \cap \bigcup_{\gamma \in \mathbb{Z}^d} Q + \gamma = \bigcup_{\gamma \in \mathbb{Z}^d} E \cap (Q+\gamma)$$

Since $f \geq 2$ on E, there exist $\gamma_1, \gamma_2 \in \mathbb{Z}^d$ with $\gamma_1 \neq \gamma_2$ such that

$$\mathfrak{m}\big(\underbrace{E \cap (\gamma_1 + Q) \cap (\gamma_2 + Q)}_{:=F}\big) > 0.$$

Then

$$F - \gamma_1 = (E - \gamma_1) \cap Q \cap [(\gamma_2 - \gamma_1) + Q],$$

so that

$$F - \gamma_1 \subset Q \cap [(\gamma_2 - \gamma_1) + Q].$$

Since $\mathfrak{m}(F - \gamma_1) = \mathfrak{m}(F) > 0$, this contradicts our hypothesis. Thus $f \leq 1$ a.e.. Since the $\mathbb{Z}^d$-translates of Q cover $\mathbb{R}^d$, $f \geq 1$ everywhere; hence $f = 1$ a.e.. Equation (2) then gives us $\mathfrak{m}(Q) = 1$. $\square$

We now prove Proposition 1.4.

Proof. We let $\{\Gamma_1, \dots, \Gamma_n\}$ be representatives of the cosets of $A(\mathbb{Z}^d)$ in $\mathbb{Z}^d$. We have

$$\mathbb{Z}^d = \bigcup_{i=1}^{n} \Gamma_i + A(\mathbb{Z}^d) = \bigcup_{i=1}^{n} \bigcup_{\gamma \in \mathbb{Z}^d} \Gamma_i + A(\gamma).$$

Thus

$$\mathbb{R}^d = [0,1]^d + \mathbb{Z}^d = \bigcup_{i=1}^{n} \bigcup_{\gamma \in \mathbb{Z}^d} [0,1]^d + \Gamma_i + A(\gamma). \tag{3}$$

We define

$$W := \bigcup_{i=1}^{n} [0,1]^d + \Gamma_i.$$

Then $\mathfrak{m}(W) = n$, and from (3) we have $\mathbb{R}^d = W + A\mathbb{Z}^d$. Therefore, since

$$\mathfrak{m}(A^{-1}W) = |\det A^{-1}| \mathfrak{m}(W) = \frac{1}{|\det A|} \mathfrak{m}(W),$$

it suffices to show that $\mathfrak{m}(A^{-1}W) = 1$. Since the determinant of A is nonzero, A is nonsingular, so

$$A^{-1}W + \mathbb{Z}^d = A^{-1}\mathbb{R}^d = \mathbb{R}^d;$$

i.e., the $\mathbb{Z}^d$-translates of $A^{-1}W$ cover $\mathbb{R}^d$. Thus, if we can show that $A^{-1}W \cap A^{-1}W + \gamma$ is null whenever $\gamma \neq 0$, we are done. Since $m([0,1]^d) = 1$ and the $\mathbb{Z}^d$-translates of $[0,1]^d$ cover $\mathbb{R}^d$, the lemma tells us that

$$m\big([0,1]^d \cap ([0,1]^d + \gamma)\big) = 0 \quad \text{for all } \gamma \neq 0.$$

Now

$$W \cap W + A\gamma = \Big[\bigcup_{i=1}^{n} [0,1]^d + \Gamma_i\Big] \cap \Big[\bigcup_{j=1}^{n} [0,1]^d + \Gamma_j + A\gamma\Big] \tag{4}$$

If $i = j$ and $\gamma \neq 0$, then clearly $\Gamma_i \neq \Gamma_j + A\gamma$. On the other hand, if $i \neq j$ and $\gamma \neq 0$, then Γ_i, Γ_j are—by definition—representatives of different cosets of $A\mathbb{Z}^d$, whence $\Gamma_i \neq \Gamma_j + A\gamma$. In either case, we have

$$m\Big(\big[[0,1]^d + \Gamma_i\big] \cap \big[[0,1]^d + \Gamma_j + A\gamma\big]\Big) = 0.$$

It follows from (4) that

$$m\big(W \cap (W + A\gamma)\big) = 0 \quad \text{whenever } \gamma \neq 0.$$

Now since A is nonsingular, it is a bijection on $\mathbb{R}^d$ and hence it preserves intersections, so that

$$m\big(A^{-1}W \cap (A^{-1}W + \gamma)\big) = m\big(A^{-1}(W \cap (W + A\gamma))\big) = 0,$$

as desired.

$\square$

Definition 1.6. A sequence of vectors $\{x_j\}_{j \in J}$ in a Hilbert space $\mathscr{H}$ is called a *Riesz sequence* if there exist constants $0 < c \leq C$ such that

$$c\Big(\sum_{j \in J} |a_j|^2\Big)^{1/2} \leq \Big\|\sum_{j \in J} a_j x_j\Big\| \leq C\Big(\sum_{j \in J} |a_j|^2\Big)^{1/2} \tag{5}$$

for all sequences of scalars $\{a_j\}_{j \in J}$. If, in addition, we have $\bigvee_{j \in J} x_j = \mathscr{H}$, where $\bigvee$ denotes the closed linear span, then we call $\{x_j\}$ a *Riesz basis*.

We note that the above definition contains a minor abuse of terminology, since the "sequences" are functions whose domains may not be $\mathbb{N}$, exactly. We now record some standard results pertaining to L^2.

Lemma 1.7. The exponential functions $\{e^{\imath\gamma\cdot x}\}_{\gamma\in\mathbb{Z}^d}$ are orthonormal in $L^2([0,2\pi]^d)$.

Proof. We compute

$$\begin{aligned}\langle e^{\imath\gamma_1\cdot x}, e^{\imath\gamma_2\cdot x}\rangle &= \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} e^{\imath\gamma_1\cdot x}e^{-\imath\gamma_2\cdot x}\,\mathrm{d}x \\ &= \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} e^{\imath(\gamma_1-\gamma_2)\cdot x}\,\mathrm{d}x.\end{aligned}$$

If $\gamma_1 = \gamma_2$ then this is clearly equal to 1. If $\gamma_1 \neq \gamma_2$, then we write the scalar product

$$(\gamma_1-\gamma_2)\cdot x = \sum_{i=1}^d (\gamma_1-\gamma_2)_i x_i,$$

and by Fubini's theorem we may write the integral above as an iterated integral:

$$\begin{aligned}&\int_{[0,2\pi]^d} e^{\imath(\gamma_1-\gamma_2)\cdot x}\,\mathrm{d}m(x) \\ &\quad= \int_0^{2\pi}\cdots\int_0^{2\pi} e^{\sum_{i=1}^d \imath(\gamma_1-\gamma_2)_i x_i}\,\mathrm{d}m(x_1),\ldots,\,\mathrm{d}m(x_d) \\ &= \prod_{i=1}^d \int_0^{2\pi} e^{\imath(\gamma_1-\gamma_2)_i x_i}\,\mathrm{d}m(x_i).\end{aligned}$$

Since $\gamma_1 \neq \gamma_2$, there must be some component i for which $(\gamma_1-\gamma_2)_i \neq 0$, and hence the integral will be 0. $\square$

In fact, more is true: $\{e^{\imath\gamma\cdot x}\}$ is an orthonormal basis for $L^2[0,2\pi]^d$. This is a very important fact, and is a corollary of the Stone-Weierstrass theorem. Recall this theorem:

Theorem 1.8 (Stone-Weierstrass/Theorem 8.1 in Conway)**.** If X is compact and $\mathscr{A}$ is a closed subalgebra of $C(X)$ such that

(i) the constant function 1 is in $\mathscr{A}$;

(ii) $\mathscr{A}$ separates points, i.e., if $x, y \in X$ and $x \neq y$, then for some $f \in \mathscr{A}$, $f(x) \neq f(y)$;

(iii) $\mathscr{A}$ is closed under complex conjugation,

then $\mathscr{A} = C(X)$. This could be rephrased by saying that if $\mathscr{A}$ is a subalgebra of $C(X)$ satisfying the above properties, then $\mathscr{A}$ is dense in $C(X)$ in the sense that for each $f \in C(X)$, there exists a sequence $\{f_n\}$ in $\mathscr{A}$ converging uniformly on X to f.

Theorem 1.9. $\{\mathrm{e}^{\imath\gamma\cdot x}\}$ is an orthonormal basis for $L^2[0, 2\pi]^d$.

Proof. For each $\gamma \in \mathbb{Z}^d$, define e_γ on $C\big(([0, 2\pi]/\{0, 2\pi\})^d\big)$ by $\mathrm{e}_\gamma(t) = \mathrm{e}^{\imath\gamma\cdot x}$. Then the algebra $\mathscr{A}$ of finite linear combinations of the e_γ (the algebra of trigonometric polynomials) satisfies the properties in the hypotheses of the Stone-Weierstrass theorem. It follows that each $f \in C\big(([0, 2\pi]/\{0, 2\pi\})^d\big)$ is a uniform limit of elements of $\mathscr{A}$. Now we may regard any $f \in C\big(([0, 2\pi]/\{0, 2\pi\})^d\big)$ as an element of $L^2[0, 2\pi]^d$, and since $L^2[0, 2\pi]^d$ is a finite measure-space, we have by Theorem 7.1 in [3] that any such f is an L^2 limit of points in $\mathscr{A}$. Finally, because $C[0, 2\pi]^d$ is dense in $L^2[0, 2\pi]^d$, the conclusion follows. $\square$

Recall that for a function f, its associated Fourier series is

$$f \sim \sum_{\gamma\in\mathbb{Z}^d} \hat{f}(\gamma)\mathrm{e}^{\imath\gamma\cdot x},$$

where the *Fourier coefficient* $\hat{f}(\gamma)$ is given by

$$\hat{f}(\gamma) = \langle f, \mathrm{e}^{\imath\gamma\cdot x}\rangle = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} f(x)\mathrm{e}^{-\imath\gamma\cdot x}\,\mathrm{d}\boldsymbol{m}(x).$$

When $f \in L^2[0, 2\pi]^d$, the Fourier series associated with f converges to f in L^2, because $\{\mathrm{e}^{\imath\gamma\cdot x}\}$ is an orthonormal basis for $L^2[0, 2\pi]^d$; cf. Theorem 4.13 in [4].

More generally, we can consider the Hilbert space $L^2([0,\ell]^d)$. In this case inner product is

$$\langle f, g\rangle = \frac{1}{\ell^d}\int_{[0,\ell]^d} f(x)\overline{g(x)}\,\mathrm{d}m(x),$$

and an argument identical to the one given for the $L^2([0,2\pi]^d)$ case shows that an orthonormal basis is give by the functions

$$\{\mathrm{e}^{\left(\frac{2\pi}{\ell}\right)\imath\gamma\cdot x} : \gamma \in \mathbb{Z}^d\}.$$

Thus, we can form the Fourier series for an element $f \in L^2[0,\ell]^d$:

$$f \sim \sum_{\gamma\in\mathbb{Z}^d} \hat{f}(\gamma)\mathrm{e}^{\left(\frac{2\pi}{\ell}\right)\imath\gamma\cdot x},$$

where the Fourier coefficients $\hat{f}(\gamma)$ are given by

$$\hat{f}(\gamma) = \langle f, \mathrm{e}^{\left(\frac{2\pi}{\ell}\right)\imath\gamma\cdot x}\rangle = \frac{1}{\ell^d}\int_{[0,\ell]^d} f(x)\mathrm{e}^{-\left(\frac{2\pi}{\ell}\right)\imath\gamma\cdot x}\,\mathrm{d}m(x).$$

Lemma 1.10. Let $\Phi \in L^2(\mathbb{R}^d)$ and let $\{a_\gamma\}_{\gamma\in\mathbb{Z}^d}$ be a sequence of scalars. We have

$$\Big\|\sum_{\gamma\in\mathbb{Z}^d} a_\gamma\Phi(x-\gamma)\Big\|^2_{L^2(\mathbb{R}^d)} = \int_{[0,2\pi]^d}\Big|\sum_{\gamma\in\mathbb{Z}^d} a_\gamma\mathrm{e}^{-\imath\gamma\cdot\xi}\Big|^2\sum_{l\in\mathbb{Z}^d}|\hat{\Phi}(\xi+2\pi l)|^2\,\mathrm{d}m(\xi). \tag{6}$$

Proof. Using Plancherel's theorem and the fact that translation corresponds to multiplication by exponentials under the Fourier transform, we have

$$\begin{aligned}\Big\|\sum_{\gamma\in\mathbb{Z}^d} a_\gamma\Phi(x-\gamma)\Big\|^2_{L^2(\mathbb{R}^d)} &= \Big\|\sum_{\gamma\in\mathbb{Z}^d} a_\gamma[\Phi(\xi-\gamma)]^\wedge\Big\|^2_{L^2(\mathbb{R}^d)}\\ &= \Big\|\sum_{\gamma\in\mathbb{Z}^d} a_\gamma\mathrm{e}^{-\imath\gamma\cdot\xi}\widehat{\Phi}(\xi)\Big\|^2_{L^2(\mathbb{R}^d)}\\ &= \int_{\mathbb{R}^d}\Big|\sum_{\gamma\in\mathbb{Z}^d} a_\gamma\mathrm{e}^{-\imath\gamma\cdot\xi}\Big|^2\big|\widehat{\Phi}(\xi)\big|^2\,\mathrm{d}m(\xi). \end{aligned}\tag{7}$$

Splitting up $\mathbb{R}^d$ into cubes and using the $2\pi\mathbb{Z}^d$-periodicity of $\mathrm{e}^{-\imath m\cdot x}$, we get that (7) equals

$$\begin{aligned}\int_{[0,2\pi]^d} \sum_{l\in\mathbb{Z}^d} \Big| \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \mathrm{e}^{-\imath\gamma\cdot(\xi+2\pi l)} \Big|^2 \big|\widehat{\Phi}(\xi+2\pi l)\big|^2 \,\mathrm{d}m(\xi) \\ = \int_{[0,2\pi]^d} \Big| \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \mathrm{e}^{-\imath\gamma\cdot\xi} \Big|^2 \sum_{l\in\mathbb{Z}^d} \big|\widehat{\Phi}(\xi+2\pi l)\big|^2 \,\mathrm{d}m(\xi).\end{aligned}$$

□

We can now prove the following standard result about Riesz sequences.

Proposition 1.11. Let Φ be a function in $L^2(\mathbb{R}^d)$. Then $\{T_\gamma\Phi\}_{\gamma\in\mathbb{Z}^d}$ is a Riesz sequence with constants c and C if and only if

$$c^2(2\pi)^{-d} \le \sum_{l\in\mathbb{Z}^d} |\hat{\Phi}(\xi+2\pi l)|^2 \le C^2(2\pi)^{-d} \quad \text{a.e. } \xi\in\mathbb{R}^d. \tag{8}$$

Proof. Assume that (8) holds; then it follows from (6) that

$$\begin{aligned}c^2(2\pi)^{-d} \int_{[0,2\pi]^d} \Big| \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \mathrm{e}^{-\imath\gamma\cdot\xi} \Big|^2 \,\mathrm{d}m(\xi) \\ \le \Big\| \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \Phi(x-\gamma) \Big\|^2_{L^2(\mathbb{R}^d)} \le C^2(2\pi)^{-d} \int_{[0,2\pi]^d} \Big| \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \mathrm{e}^{-\imath\gamma\cdot\xi} \Big|^2 \,\mathrm{d}m(\xi).\end{aligned}$$

Now since $\{\mathrm{e}^{\imath\gamma\cdot\xi}\}_{\gamma\in\mathbb{Z}^d}$ is an orthonormal family in $L^2([0,2\pi]^d)$, we have

$$\int_{[0,2\pi]^d} \Big| \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \mathrm{e}^{-\imath\gamma\cdot\xi} \Big|^2 \,\mathrm{d}m(\xi) = (2\pi)^d \Big\| \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \mathrm{e}^{-\imath\gamma\cdot\xi} \Big\|^2_{L^2[0,2\pi]} = (2\pi)^d \sum_{\gamma\in\mathbb{Z}^d} |a_\gamma|^2$$

by the Pythagorean theorem. Thus we have

$$c^2 \sum_{\gamma\in\mathbb{Z}^d} |a_\gamma|^2 \le \Big\| \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \Phi(x-\gamma) \Big\|^2_{L^2(\mathbb{R}^d)} \le C^2 \sum_{\gamma\in\mathbb{Z}^d} |a_\gamma|^2,$$

which shows that $\{T_\gamma\Phi\}_{\gamma\in\mathbb{Z}^d}$ is a Riesz sequence with constants c and C. Now we wish to prove the converse. We let

$$C_\alpha = \Big\{\xi\in[0,2\pi]^d : \sum_{l\in\mathbb{Z}^d} |\hat{\Phi}(\xi+2\pi l)|^2 > \alpha\Big\}.$$

The plan is to show that it is not possible for C_α to have positive measure for any $\alpha > \frac{C^2}{(2\pi)^d}$. This will establish the right hand side of (8). Suppose that for some α, C_α has positive measure. Then since the indicator function $\chi_{C_\alpha}(\xi)$ is in $L^2[0, 2\pi]^d$, it is equal to its Fourier series a.e. (by Carleson's theorem) and in L^2:

$$\chi_{C_\alpha}(\xi) = \sum_{\gamma\in\mathbb{Z}^d} \widehat{\chi_{C_\alpha}}(\gamma)\mathrm{e}^{-\imath\gamma\cdot\xi} \quad \text{a.e. and in } L^2[0, 2\pi]^d \tag{9}$$

where the coefficients

$$\widehat{\chi_{C_\alpha}}(\gamma) = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} \chi_{C_\alpha}(x)\mathrm{e}^{-\imath\gamma\cdot x}\,\mathrm{d}m(x)$$

are called the Fourier coefficients. Since the convergence of the Fourier series in (9) is in L^2 and exponentials are orthonormal, it follows from the Pythagorean theorem that the sequence of Fourier coefficients $\{\widehat{\chi_{C_\alpha}}(\gamma)\}_{\gamma\in\mathbb{Z}^d}$ is in $l^2(\mathbb{Z}^d)$. Now, applying Lemma 1.10 to the sequence $\{\widehat{\chi_{C_\alpha}}(\gamma)\}_{\gamma\in\mathbb{Z}^d}$ yields

$$\begin{aligned}
\Big\| \sum_{\gamma\in\mathbb{Z}^d} \widehat{\chi_{C_\alpha}}(\gamma)\Phi(x-\gamma)\Big\|^2_{L^2(\mathbb{R}^d)} &= \int_{[0,2\pi]^d} \Big|\chi_{C_\alpha}(\xi)\Big|^2 \sum_{l\in\mathbb{Z}^d} |\hat{\Phi}(\xi+2\pi l)|^2\,\mathrm{d}m(\xi)\\
&= \int_{C_\alpha} \sum_{l\in\mathbb{Z}^d} |\hat{\Phi}(\xi+2\pi l)|^2\,\mathrm{d}m(\xi)\\
&\geq \alpha m(C_\alpha).
\end{aligned}$$

Now by (9) and the Pythagorean theorem, we have

$$\sum_{l\in\mathbb{Z}^d} |\widehat{\chi_{C_\alpha}}(\gamma)|^2 = \|\chi_{C_\alpha}\|^2_{L^2[0,2\pi]^d} = \frac{m(C_\alpha)}{(2\pi)^d},$$

and hence

$$\Big\| \sum_{\gamma\in\mathbb{Z}^d} \widehat{\chi_{C_\alpha}}(\gamma)\Phi(x-\gamma)\Big\|^2_{L^2(\mathbb{R}^d)} \geq \alpha(2\pi)^d \sum_{l\in\mathbb{Z}^d} |\widehat{\chi_{C_\alpha}}(\gamma)|^2.$$

Now by assumption, $\{\Phi(x-\gamma)\}$ is a Riesz sequence with constants c and C, and therefore

$$\alpha(2\pi)^d \sum_{l\in\mathbb{Z}^d} |\widehat{\chi_{C_\alpha}}(\gamma)|^2 \leq C^2 \sum_{l\in\mathbb{Z}^d} |\widehat{\chi_{C_\alpha}}(\gamma)|^2.$$

Hence $\alpha \leq \frac{C^2}{(2\pi)^d}$. The left hand side of (8) is established in the same way. $\square$

Corollary 1.12. Let Φ be a function in $L^2(\mathbb{R}^d)$. Then $\{T_\gamma \Phi\}_{\gamma\in\mathbb{Z}^d}$ is an orthonormal family in $L^2(\mathbb{R}^d)$ if and only if

$$\sum_{l\in\mathbb{Z}^d} |\hat{\Phi}(\xi + 2\pi l)|^2 = (2\pi)^{-d} \quad \text{a.e..} \tag{10}$$

Proof. The preceding proposition implies that condition (10) is equivalent to having

$$\Big\| \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \Phi(x-\gamma) \Big\|^2_{L^2(\mathbb{R}^d)} = \sum_{\gamma\in\mathbb{Z}^d} |a_\gamma|^2 \tag{11}$$

for every sequence of scalars $\{a_\gamma\}_{\gamma\in\mathbb{Z}^d}$. Suppose that $\{\Phi(x-\gamma)\}$ is an orthonormal family. Then (11) holds by Pythagoras' theorem. Suppose conversely that (11) holds; then $\{\Phi(x-\gamma)\}$ is seen to be orthonormal by choosing appropriate sequences. Specifically, if we choose $\{a_\gamma\}$ to be the sequence

$$a_\gamma = \begin{cases} 1 & \text{if } \gamma = \gamma_0 \\ 0 & \text{otherwise,} \end{cases}$$

then we see that $\|\Phi(x-\gamma_0)\| = 1$; $\gamma_0 \in \mathbb{Z}^d$ was arbitrary, so $\Phi(x-\gamma)$ has unit norm for all $\gamma \in \mathbb{Z}^d$. If we choose

$$b_\gamma = \begin{cases} 1 & \text{if } \gamma = \gamma_1 \text{ or } \gamma = \gamma_2 \\ 0 & \text{otherwise,} \end{cases}$$

then

$$2 = \Big\| \sum_{\gamma\in\mathbb{Z}^d} b_\gamma \Phi(x-\gamma) \Big\|^2_{L^2(\mathbb{R}^d)} = 2 + 2\mathrm{Re}\big\langle b_{\gamma_1}\Phi(x-\gamma_1), b_{\gamma_2}\Phi(x-\gamma_2)\big\rangle,$$

whence

$$\mathrm{Re}\big\langle \Phi(x-\gamma_1), \Phi(x-\gamma_2)\big\rangle = 0.$$

If we choose

$$c_\gamma = \begin{cases} \imath & \text{if } \gamma = \gamma_1 \\ 1 & \text{if } \gamma = \gamma_2 \\ 0 & \text{otherwise,} \end{cases}$$

then we get

$$
\begin{aligned}
2 &= \Big\| \sum_{\gamma\in\mathbb{Z}^d} c_\gamma \Phi(x-\gamma) \Big\|^2_{L^2(\mathbb{R}^d)} \\
&= 2 + \big\langle \imath\Phi(x-\gamma_1), \Phi(x-\gamma_2)\big\rangle + \big\langle \Phi(x-\gamma_2), \imath\Phi(x-\gamma_1)\big\rangle,
\end{aligned}
$$

whence

$$
\imath\Big(\big\langle \Phi(x-\gamma_1), \Phi(x-\gamma_2)\big\rangle - \overline{\big\langle \Phi(x-\gamma_1), \Phi(x-\gamma_2)\big\rangle} \Big) = 0,
$$

whence

$$
\operatorname{Im}\big\langle \Phi(x-\gamma_1), \Phi(x-\gamma_2)\big\rangle = 0.
$$

$\square$

Lemma 1.13. (i) Given a multiresolution analysis $\{V_j\}_{j\in\mathbb{N}}$ with scaling function Φ, we have $f \in V_1$ if and only if there exists a $2\pi\mathbb{Z}^d$-periodic function $\mathrm{m} \in L^2[0,2\pi]^d$ such that

$$
\hat{f}(A^*\xi) = \mathrm{m}(\xi)\hat{\Phi}(\xi), \tag{12}
$$

where the symbol $\hat{}$ denotes taking the Fourier transform, and A^* denotes the adjoint of the linear operator A (so A^* is the conjugate transpose of A).

(ii) We have

$$
\int_{[0,2\pi]^d} |\mathrm{m}(\xi)|^2 \,\mathrm{d}\boldsymbol{m}(\xi) = \frac{(2\pi)^d}{q} \int_{\mathbb{R}^d} |f(x)|^2 \,\mathrm{d}\boldsymbol{m}(x),
$$

where $q = |\det(A)|$.

Proof. Since $f \in V_1$ if and only if $D_{A^{-1}}f \in V_0$, we have

$$
f(A^{-1}x) = \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \Phi(x-\gamma) \tag{13}
$$

for some sequence of scalars $\{a_\gamma\}$. Recall that for any linear operator A,

$$
(f\circ A)\widehat{\ }(\xi) = (\det A)^{-1}\hat{f}\big((A^{-1})^*(\xi)\big). \tag{14}
$$

Applying this to take the Fourier transform of both sides of (13), and using the continuity of the Fourier transform, we get

$$q\hat{f}(A^*\xi) = \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \mathrm{e}^{\imath\xi\cdot\gamma}\hat{\Phi}(\xi).$$

Thus, if we take

$$\mathrm{m}(\xi) = q^{-1}\sum_{\gamma\in\mathbb{Z}^d} a_\gamma \mathrm{e}^{\imath\xi\cdot\gamma}, \tag{15}$$

we get $\hat{f}(A^*\xi) = \mathrm{m}(\xi)\hat{\Phi}(\xi)$, which gives (12). Having defined m in this way, it follows from the Pythagorean theorem that

$$\|\mathrm{m}\|^2_{L^2[0,2\pi]^d} = \tfrac{1}{q^2}\sum_{\gamma\in\mathbb{Z}^d} |a_\gamma|^2.$$

It follows from (13), the Pythagorean theorem, and the behaviour of the Lebesgue integral under dilation that

$$\sum_{\gamma\in\mathbb{Z}^d} |a_\gamma|^2 = \int_{\mathbb{R}^d} |f(A^{-1}x)|^2\,\mathrm{d}\boldsymbol{m}(x) = q\int_{\mathbb{R}^d} |f(x)|^2\,\mathrm{d}\boldsymbol{m}(x).$$

Therefore,

$$\int_{[0,2\pi]^d} |\mathrm{m}(\xi)|^2\,\mathrm{d}\boldsymbol{m}(\xi) = \frac{(2\pi)^d}{q}\int_{\mathbb{R}^d} |f(x)|^2\,\mathrm{d}\boldsymbol{m}(x).$$

Thus, in particular, $\mathrm{m}(\xi) \in L^2[0,2\pi]^d$; note that by uniqueness of the Fourier coefficients (Riesz-Fischer), the coefficients in (15) are uniquely determined by the Fourier series expansion for m.

Suppose conversely that there exists a $2\pi\mathbb{Z}^d$-periodic function $\mathrm{m} \in L^2[0,2\pi]^d$ such that (12) holds. Then since m is square-integrable, m has Fourier series expansion

$$\mathrm{m}(\xi) = \sum_{\gamma\in\mathbb{Z}^d} \widehat{\mathrm{m}}(\gamma)\mathrm{e}^{\imath\gamma\cdot\xi} \quad \text{in } L^2 \text{ and a.e..}$$

Thus we can write

$$q\hat{f}(A^*\xi) = \sum_{\gamma\in\mathbb{Z}^d} q\widehat{\mathrm{m}}(\gamma)\mathrm{e}^{\imath\gamma\cdot\xi}\hat{\Phi}(\xi).$$

Now we can apply (14) to this to obtain

$$(f \circ A^{-1})\hat{}(\xi) = \sum_{\gamma \in \mathbb{Z}^d} q\widehat{\mathrm{m}}(\gamma)\mathrm{e}^{\imath\gamma\cdot\xi}\hat{\Phi}(\xi).$$

Now since, by the Plancherel Theorem, the Fourier transform is an isometric isomorphism on L^2, we can take the inverse Fourier transform of both sides of the above equation to obtain

$$f(A^{-1}x) = \sum_{\gamma \in \mathbb{Z}^d} q\widehat{\mathrm{m}}(\gamma)\Phi(x-\gamma).$$

Thus

$$D_{A^{-1}}(f) \in \overline{\mathrm{span}}\{T_\gamma\Phi\} = V_0,$$

whence $f \in V_1$. $\square$

With this lemma in hand, suppose that $\{V_j\}_{j\in\mathbb{N}}$ is a multiresolution analysis associated with the dilation matrix A, with scaling function Φ, and let m be the $2\pi\mathbb{Z}^d$-periodic function satisfying

$$\int_{[0,2\pi]^d} |\mathrm{m}(\xi)|^2 \,\mathrm{d}m(\xi) = \frac{(2\pi)^d}{q}\int_{\mathbb{R}^d} |f(x)|^2 \,\mathrm{d}m(x).$$

Then $\mathrm{m} \in L^2[0,2\pi]^d$, since $f \in V_1 \subset L^2(\mathbb{R}^d)$, so m has Fourier series expansion

$$\mathrm{m}(\xi) = \sum_{\gamma\in\mathbb{Z}^d} \widehat{\mathrm{m}}(\gamma)\mathrm{e}^{\imath\gamma\cdot\xi} \quad \text{in } L^2 \text{ and a.e..}$$

Now let $\{\Gamma_r + A(\mathbb{Z}^d)\}_{r=0}^{q-1}$ be the distinct cosets of $A(\mathbb{Z}^d)$ in $\mathbb{Z}^d$. Define

$$\mathrm{m}^r := \sum_{\gamma\in\Gamma_r + A(\mathbb{Z}^d)} \widehat{\mathrm{m}}(\gamma)\mathrm{e}^{\imath\gamma\cdot\xi},$$

so that

$$\mathrm{m} = \sum_{r=0}^{q-1} \mathrm{m}^r.$$

Then

$$\begin{aligned}
\mathrm{m}^r &= \sum_{\gamma \in A(\mathbb{Z}^d)} \widehat{\mathrm{m}}(\Gamma_r + \gamma) \mathrm{e}^{\imath(\gamma+\Gamma_r)\cdot\xi} \\
&= \mathrm{e}^{\imath\Gamma_r\cdot\xi} \sum_{\gamma \in A(\mathbb{Z}^d)} \widehat{\mathrm{m}}(\Gamma_r + \gamma) \mathrm{e}^{\imath\gamma\cdot\xi} \\
&= \mathrm{e}^{\imath\Gamma_r\cdot\xi} \sum_{\gamma \in \mathbb{Z}^d} \widehat{\mathrm{m}}(\Gamma_r + A\gamma) \mathrm{e}^{\imath(A\gamma)\cdot\xi} \\
&= \mathrm{e}^{\imath\Gamma_r\cdot\xi} \sum_{\gamma \in \mathbb{Z}^d} \widehat{\mathrm{m}}(\Gamma_r + A\gamma) \mathrm{e}^{\imath\gamma\cdot(A^*\xi)}.
\end{aligned}$$

Defining

$$\mu^r(\xi) = \sum_{\gamma \in \mathbb{Z}^d} \widehat{\mathrm{m}}(\Gamma_r + A\gamma) \mathrm{e}^{\imath\gamma\cdot\xi},$$

we have

$$\mathrm{m}^r(\xi) = \mathrm{e}^{\imath\Gamma_r\cdot\xi} \mu^r(A^*\xi). \tag{16}$$

Note that the $\mu_f^r(\xi)$ are measurable: the functions $\widehat{\mathrm{m}}(\Gamma_r + A\gamma)\mathrm{e}^{\imath\gamma\cdot\xi}$ are measurable, and hence finite sums of these are measurable. Now since $\mathbb{Z}^d$ is countable, we can reindex it by $n \leftrightarrow \gamma_n$ for each $n \in \mathbb{N}$. Then

$$\mu^r(\xi) = \sum_{\gamma \in \mathbb{Z}^d} \widehat{\mathrm{m}}(\Gamma_r + A\gamma) \mathrm{e}^{\imath\gamma\cdot\xi} = \lim_{N\to\infty} \sum_{n=1}^{N} \widehat{\mathrm{m}}(\Gamma_r + A\gamma_n) \mathrm{e}^{\imath\gamma_n\cdot\xi}$$

is a pointwise limit of measurable functions. Moreover, the $\mu^r(\xi)$ are in $L^2[0, 2\pi]^d$, which follows from (16) and the fact that the functions m^r are in $L^2[0, 2\pi]^d$, because

$$\|\mathrm{m}\|_{L^2}^2 = \sum_{r=0}^{q-1} \|\mathrm{m}^r\|_{L^2}^2.$$

Finally, since a sum of periodic functions is periodic, the functions $\mu^r(\xi)$ are $2\pi\mathbb{Z}^d$-periodic, being the sum of the $2\pi\mathbb{Z}^d$-periodic functions $\widehat{\mathrm{m}}(\Gamma_r + A\gamma)\mathrm{e}^{\imath\gamma\cdot\xi}$.

From the above remarks we see that:

There is a one-to-one correspondence between functions $f \in V_1$ and q-tuples of $2\pi\mathbb{Z}^d$-periodic functions $\mu^r(\xi)$ whose restrictions to $[0, 2\pi]^d$ are in $L^2[0, 2\pi]^d$:

$$f \in V_1 \leftrightarrow \begin{bmatrix} \mu^0(\xi) & \mu^1(\xi) & \cdots & \mu^{q-1}(\xi) \end{bmatrix}, \tag{17}$$

given by

$$\hat{f}(A^*\xi) = \sum_{r=0}^{q-1} \mathrm{e}^{\imath \Gamma_r \cdot \xi} \mu^r(A^*\xi)\hat{\Phi}(\xi). \tag{18}$$

Now suppose f_1, f_2 are two elements of V_1. Using Plancherel's theorem we can express the inner product $\langle T_{\gamma_1} f_1, T_{\gamma_2} f_2 \rangle$ of the $\mathbb{Z}^d$-translates of f_1, f_2 in terms of the inner product of the corresponding functions $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$, where

$$\boldsymbol{\mu}_i(\xi) := \begin{bmatrix} \mu_i^0(\xi) & \mu_i^1(\xi) & \cdots & \mu_i^{q-1}(\xi) \end{bmatrix}^T, \quad i = 1, 2. \tag{19}$$

We have, by Plancherel's theorem,

$$\begin{aligned} \langle T_{\gamma_1} f_1, T_{\gamma_2} f_2 \rangle &= \langle \widehat{T_{\gamma_1} f_1}, \widehat{T_{\gamma_2} f_2} \rangle \\ &= \int_{\mathbb{R}^d} \widehat{T_{\gamma_1} f_1}(\xi) \overline{\widehat{T_{\gamma_1} f_2}(\xi)} \, \mathrm{d}m(\xi) \\ &= \int_{\mathbb{R}^d} \mathrm{e}^{-\imath \gamma_1 \cdot \xi} \widehat{f_1}(\xi) \overline{\mathrm{e}^{-\imath \gamma_2 \cdot \xi} \widehat{f_2}(\xi)} \, \mathrm{d}m(\xi) \\ &= \int_{\mathbb{R}^d} \widehat{f_1}(\xi) \overline{\widehat{f_2}(\xi)} \mathrm{e}^{-\imath (\gamma_1 - \gamma_2) \cdot \xi} \, \mathrm{d}m(\xi). \end{aligned}$$

Now by part (i) of Lemma 1.13 we have $2\pi\mathbb{Z}^d$-periodic functions m_1 and m_2 such that

$$\widehat{f_1}(A^*\xi) = \mathrm{m}_1(\xi)\hat{\Phi}(\xi) \quad \text{and} \quad \widehat{f_2}(\xi) = \mathrm{m}_2(\xi)\hat{\Phi}(\xi).$$

Using the behaviour of the Lebesgue integral under translation, the above inner product becomes

$$\begin{aligned} &|\det(A^*)| \int_{\mathbb{R}^d} \widehat{f_1}(A^*\xi) \overline{\widehat{f_2}(A^*\xi)} \mathrm{e}^{-\imath (\gamma_1 - \gamma_2) \cdot (A^*\xi)} \, \mathrm{d}m(\xi) \\ &= q \int_{\mathbb{R}^d} \mathrm{m}_1(\xi) \overline{\mathrm{m}_2(\xi)} |\hat{\Phi}(\xi)|^2 \mathrm{e}^{-\imath (\gamma_1 - \gamma_2) \cdot (A^*\xi)} \, \mathrm{d}m(\xi), \end{aligned}$$

where $q = |\det A|$. Now since m_1, m_2, and $\mathrm{e}^{\imath\gamma\xi}$ are $2\pi\mathbb{Z}^d$-periodic, we can rewrite this as

$$\begin{aligned} q \int_{[0,2\pi]^d} & \sum_{\gamma\in\mathbb{Z}^d} \mathrm{m}_1(\xi+2\pi\gamma)\overline{\mathrm{m}_2(\xi+2\pi\gamma)}|\hat{\Phi}(\xi+2\pi\gamma)|^2\, \mathrm{e}^{-\imath A(\gamma_1-\gamma_2)\cdot(\xi+2\pi\gamma)}\, \mathrm{d}m(\xi) \\ &= q\int_{[0,2\pi]^d} \mathrm{m}_1(\xi)\overline{\mathrm{m}_2(\xi)}\, \mathrm{e}^{-\imath A(\gamma_1-\gamma_2)\cdot\xi} \sum_{\gamma\in\mathbb{Z}^d} |\hat{\Phi}(\xi+2\pi\gamma)|^2\, \mathrm{d}m(\xi). \end{aligned}$$

Now since Φ is a scaling function, we have by Corollary 1.12 that $\sum_{\gamma\in\mathbb{Z}^d}|\hat{\Phi}(\xi+2\pi\gamma)|^2 = (2\pi)^{-d}$, whence the above integral becomes

$$\begin{aligned} &\frac{q}{(2\pi)^d}\int_{[0,2\pi]^d} \mathrm{m}_1(\xi)\overline{\mathrm{m}_2(\xi)}\, \mathrm{e}^{-\imath A(\gamma_1-\gamma_2)\cdot\xi}\, \mathrm{d}m(\xi) \\ &= \frac{q}{(2\pi)^d}\int_{[0,2\pi]^d} \mathrm{e}^{-\imath A(\gamma_1-\gamma_2)\cdot\xi}\Big(\sum_{r=0}^{q-1}\mathrm{m}_1^r(\xi)\Big)\overline{\Big(\sum_{r=0}^{q-1}\mathrm{m}_2^r(\xi)\Big)}\, \mathrm{d}m(\xi). \end{aligned}$$

Now recall that

$$\mathrm{m}_1^{r_1}(\xi) = \mathrm{e}^{\imath\Gamma_{r_1}\cdot\xi}\mu_1^{r_1}(A^*\xi),$$

and

$$\mathrm{m}_2^{r_2}(\xi) = \mathrm{e}^{\imath\Gamma_{r_2}\cdot\xi}\mu_2^{r_2}(A^*\xi).$$

Since the exponential functions form an orthonormal family, our inner product becomes

$$\begin{aligned} &\frac{q}{(2\pi)^d}\int_{[0,2\pi]^d} \mathrm{e}^{-\imath(\gamma_1-\gamma_2)\cdot(A^*\xi)}\sum_{r=0}^{q-1}\mu_1^r(A^*\xi)\overline{\mu_2^r(A^*\xi)}\, \mathrm{d}m(\xi) \\ &= \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} \mathrm{e}^{-\imath(\gamma_1-\gamma_2)\cdot\xi}\sum_{r=0}^{q-1}\mu_1^r(\xi)\overline{\mu_2^r(\xi)}\, \mathrm{d}m(\xi) \\ &= \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} \mathrm{e}^{-\imath(\gamma_1-\gamma_2)\cdot\xi}\big\langle \boldsymbol{\mu}_1(\xi), \boldsymbol{\mu}_2(\xi)\big\rangle_{\mathbb{C}^q}\, \mathrm{d}m(\xi), \end{aligned}$$

where the $\boldsymbol{\mu}_i$ are defined by (19). So altogether, we have

$$\big\langle T_{\gamma_1}f_1, T_{\gamma_2}f_2\big\rangle_{L^2(\mathbb{R}^d)} = \Big\langle \big\langle \boldsymbol{\mu}_1(\xi), \boldsymbol{\mu}_2(\xi)\big\rangle_{\mathbb{C}^q}, \mathrm{e}^{\imath(\gamma_1-\gamma_2)\cdot\xi}\Big\rangle_{L^2[0,2\pi]^d}.$$

In other words, $\langle T_{\gamma_1} f_1, T_{\gamma_2} f_2 \rangle$ is the Fourier coefficient of the complex scalar inner product $\boldsymbol{\mu}_1(\xi) \cdot \boldsymbol{\mu}_2(\xi)$ at $\gamma_1 - \gamma_2$.

Theorem 1.14 (Proposition 5.9 in [8])**.** Let $f_0, \dots, f_{q-1} \in V_1$. With the notation above, we have

(i) The family $\{f_0(x-\gamma)\}_{\gamma \in \mathbb{Z}^d}$ is orthonormal if and only if

$$\sum_{r=0}^{q-1} |\mu_0^r(\xi)|^2 = 1 \qquad \text{a.e.}; \tag{20}$$

(ii) For $0 \le s \le q-1$, the family $\{f_j(x-\gamma)\}_{\gamma \in \mathbb{Z}^d; j=0,\dots,s}$ is orthonormal if and only if the vectors

$$\boldsymbol{\mu}_j(\xi) := \begin{bmatrix} \mu_j^0(\xi) & \mu_j^1(\xi) & \cdots & \mu_j^{q-1}(\xi) \end{bmatrix}^T \tag{21}$$

are mutually orthonormal in $\mathbb{C}^q$ for $j = 0, \dots, s$ and for a.e. $\xi \in [0, 2\pi]^d$.

(iii) The family $\{f_j(x-\gamma)\}_{\gamma \in \mathbb{Z}^d; j=0,\dots,q-1}$ is an orthonormal basis for V_1 if and only if the matrix

$$\mathbf{U}(\xi) := \begin{bmatrix} \mu_0^0(\xi) & \mu_0^1(\xi) & \cdots & \mu_0^{q-1}(\xi) \\ \mu_1^0(\xi) & \mu_1^1(\xi) & \cdots & \mu_1^{q-1}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{q-1}^0(\xi) & \mu_{q-1}^1(\xi) & \cdots & \mu_{q-1}^{q-1}(\xi) \end{bmatrix} \tag{22}$$

is unitary for a.e. $\xi \in [0, 2\pi]^d$. In particular, V_1 has dimension $q = |\det A|$.

(iv) If Φ is the scaling function associated with the MRA $\{V_j\}_{j \in \mathbb{Z}}$, then we can find measurable, $2\pi\mathbb{Z}^d$-periodic functions $\{\mu_j^r\}_{j=1,\dots,q-1; r=0,\dots,q-1}$, whose restrictions to $[0, 2\pi]^d$ are in $L^2[0, 2\pi]^d$, such that the matrix

$$\mathbf{U}(\xi) := \begin{bmatrix} \mu_\Phi^0(\xi) & \mu_\Phi^1(\xi) & \cdots & \mu_\Phi^{q-1}(\xi) \\ \mu_1^0(\xi) & \mu_1^1(\xi) & \cdots & \mu_1^{q-1}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{q-1}^0(\xi) & \mu_{q-1}^1(\xi) & \cdots & \mu_{q-1}^{q-1}(\xi) \end{bmatrix} \tag{23}$$

whose first row is obtained from Φ via (17), is unitary. We thereby obtain a collection $\{\Phi, \Psi_1, \ldots, \Psi_{q-1}\}$ of elements whose $\mathbb{Z}^d$-translates form an orthonormal basis for V_1, with Φ being one of the basis elements.

Proof. (i) Suppose first that

$$\sum_{r=0}^{q-1} |\mu_0^r(\xi)|^2 = 1 \quad \text{a.e..}$$

Writing

$$\boldsymbol{\mu}_0(\xi) := \begin{bmatrix} \mu_0^0(\xi) & \mu_0^1(\xi) & \cdots & \mu_0^{q-1}(\xi), \end{bmatrix}^T$$

this says that

$$\|\boldsymbol{\mu}_0(\xi)\|^2 = 1 \quad \text{a.e..}$$

Now let $\gamma_1, \gamma_2 \in \mathbb{Z}^d$. If $\gamma_1 = \gamma_2$, then

$$\begin{aligned} \left\langle T_{\gamma_1} f_0, T_{\gamma_2} f_0 \right\rangle &= \left\langle \|\boldsymbol{\mu}_0(\xi)\|^2, \mathrm{e}^{\imath(\gamma_1 - \gamma_2)\cdot\xi} \right\rangle \\ &= \left\langle 1, \mathrm{e}^0 \right\rangle \\ &= \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} 1 \, \mathrm{d}m(\xi) \\ &= 1. \end{aligned}$$

If $\gamma_1 \neq \gamma_2$, then

$$\begin{aligned} \left\langle T_{\gamma_1} f_0, T_{\gamma_2} f_0 \right\rangle &= \left\langle 1, \mathrm{e}^{\imath(\gamma_1 - \gamma_2)\cdot\xi} \right\rangle \\ &= \left\langle 1, \mathrm{e}^{\imath\gamma_1\cdot\xi} \overline{\mathrm{e}^{\imath\gamma_2\cdot\xi}} \right\rangle \\ &= \left\langle \mathrm{e}^{\imath\gamma_2\cdot\xi}, \mathrm{e}^{\imath\gamma_1\cdot\xi} \right\rangle \\ &= 0. \end{aligned}$$

Thus if $\|\boldsymbol{\mu}_0(\xi)\|^2 = 1$ a.e., then $\{T_\gamma f_0\}_{\gamma\in\mathbb{Z}^d}$ is an orthonormal collection. On the other hand, if the $2\pi\mathbb{Z}^d$-translates of f_0 form an orthonormal collection, then

$$\left\langle \|\boldsymbol{\mu}_0(\xi)\|^2, \mathrm{e}^{\imath(\gamma_1-\gamma_2)\cdot\xi} \right\rangle = \left\langle T_{\gamma_1} f_0, T_{\gamma_2} f_0 \right\rangle = \begin{cases} 1 & \text{if } \gamma_1 = \gamma_2 \\ 0 & \text{otherwise.} \end{cases}$$

From this it follows that

$$\left\langle \|\boldsymbol{\mu}_0(\xi)\|^2, \mathrm{e}^{\imath\gamma\cdot\xi} \right\rangle = \begin{cases} 1 & \text{if } \gamma = \mathbf{0} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, all the Fourier coefficients of $\|\boldsymbol{\mu}_0(\xi)\|^2$ are zero, except for the coefficient at $\mathbf{0}$, which is 1; in particular, the sequence of Fourier coefficients is in $l^2(\mathbb{Z}^d)$. Now since the Fourier transform is an isometric isomorphism from $L^2[0,2\pi]^d$ onto $l^2(\mathbb{Z}^d)$, there exists a unique function $G \in L^2[0,2\pi]^d$ such that

$$\widehat{G}(\gamma) = \widehat{\|\boldsymbol{\mu}_0\|^2}(\gamma) \quad \text{for all } \gamma \in \mathbb{Z}^d,$$

and by Carleson's theorem G converges in L^2 and a.e. to its Fourier series:

$$G(\xi) = \sum_{\gamma\in\mathbb{Z}^d} \widehat{G}(\gamma)\mathrm{e}^{\imath\gamma\cdot\xi} = \sum_{\gamma\in\mathbb{Z}^d} \widehat{\|\boldsymbol{\mu}_0\|^2}(\gamma)\mathrm{e}^{\imath\gamma\cdot\xi} = 1 \quad \text{a.e..}$$

We also see from this that $G \in L^1[0,2\pi]^d$. Now observe that $\|\boldsymbol{\mu}_0(\xi)\|^2$ is in $L^1[0,2\pi]^d$ as well, since

$$1 = \left\langle T_\gamma f_0, T_\gamma f_0 \right\rangle = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \|\boldsymbol{\mu}_0(\xi)\|^2 \,\mathrm{d}m(\xi).$$

Since the Fourier coefficients for G are equal to the Fourier coefficients for $\|\boldsymbol{\mu}_0(\xi)\|^2$, we have from Corollary 8.27 in Folland [5] that $G = \|\boldsymbol{\mu}_0(\xi)\|^2$ a.e.. Therefore $\|\boldsymbol{\mu}_0(\xi)\|^2$ is in $L^2[0,2\pi]^d$ as well, so it converges a.e. to its Fourier series:

$$\|\boldsymbol{\mu}_0(\xi)\|^2 = \sum_{\gamma\in\mathbb{Z}^d} \left\langle \|\boldsymbol{\mu}_0(\xi)\|^2, \mathrm{e}^{\imath\gamma\cdot\xi} \right\rangle \mathrm{e}^{\imath\gamma\cdot\xi} = 1 \quad \text{a.e..}$$

(ii) The argument is similar to that of (i). Suppose first that the vectors

$$\boldsymbol{\mu}_j(\xi) := \begin{bmatrix} \mu_j^0(\xi) & \mu_j^1(\xi) & \cdots & \mu_j^{q-1}(\xi) \end{bmatrix}^T \tag{24}$$

are mutually orthonormal in $\mathbb{C}^q$ for $j = 0, \ldots, s$ and for a.e. $\xi \in [0,2\pi]^d$. We wish to show that for $0 \le s \le q-1$, the family $\{T_\gamma f_j\}_{\gamma\in\mathbb{Z}^d; j=0,\ldots,s}$ is orthonormal. Now let

$\gamma_1, \gamma_2 \in \mathbb{Z}^d$, and let $i, j \in \{0, \ldots, s\}$. If $\gamma_1 = \gamma_2 = \gamma$ and $i = j$, then

$$\begin{aligned}
\left\langle T_{\gamma_1} f_i, T_{\gamma_2} f_j \right\rangle_{L^2(\mathbb{R}^d)} &= \left\langle \boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi), \mathrm{e}^{\imath(\gamma_1 - \gamma_2)\cdot\xi} \right\rangle_{L^2[0,2\pi]^d}. \\
&= \left\langle \|\boldsymbol{\mu}_i(\xi)\|^2, \mathrm{e}^{\imath(\gamma - \gamma)\cdot\xi} \right\rangle_{L^2[0,2\pi]^d}. \\
&= \left\langle 1, \mathrm{e}^0 \right\rangle \\
&= 1.
\end{aligned}$$

If $i = j$ but $\gamma_1 \neq \gamma_2$, then

$$\begin{aligned}
\left\langle T_{\gamma_1} f_i, T_{\gamma_2} f_j \right\rangle_{L^2(\mathbb{R}^d)} &= \left\langle \|\boldsymbol{\mu}_i(\xi)\|^2, \mathrm{e}^{\imath(\gamma_1 - \gamma_2)\cdot\xi} \right\rangle \\
&= \left\langle 1, \mathrm{e}^{\imath\gamma_1\cdot\xi} \overline{\mathrm{e}^{\imath\gamma_2\cdot\xi}} \right\rangle \\
&= \left\langle \mathrm{e}^{\imath\gamma_2\cdot\xi}, \mathrm{e}^{\imath\gamma_1\cdot\xi} \right\rangle \\
&= 0.
\end{aligned}$$

If $i \neq j$, then

$$\begin{aligned}
\left\langle T_{\gamma_1} f_i, T_{\gamma_2} f_j \right\rangle_{L^2(\mathbb{R}^d)} &= \left\langle \boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi), \mathrm{e}^{\imath(\gamma_1 - \gamma_2)\cdot\xi} \right\rangle_{L^2[0,2\pi]^d}. \\
&= \left\langle \boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi), \mathrm{e}^{\imath(\gamma_1 - \gamma_2)\cdot\xi} \right\rangle \\
&= \left\langle 0, \mathrm{e}^{\imath(\gamma_1 - \gamma_2)\cdot\xi} \right\rangle \\
&= 0.
\end{aligned}$$

Thus if the vectors (24) are mutually orthonormal for $j = 0, \ldots, s$, the family $\{T_\gamma f_j\}_{\gamma\in\mathbb{Z}^d; j=0,\ldots,s}$ are orthonormal.

Suppose, conversely, that the family $\{f_j(x - \gamma)\}_{\gamma\in\mathbb{Z}^d; j=0,\ldots,s}$ is orthonormal; we wish to show that the vectors

$$\boldsymbol{\mu}_j(\xi) := \begin{bmatrix} \mu_j^0(\xi) & \mu_j^1(\xi) & \cdots & \mu_j^{q-1}(\xi) \end{bmatrix}^T \tag{25}$$

are mutually orthonormal in $\mathbb{C}^q$ for $j = 0, \ldots, s$ and for a.e. $\xi \in [0, 2\pi]^d$. We have

$$\left\langle \boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi), \mathrm{e}^{\imath(\gamma_1 - \gamma_2)\cdot\xi} \right\rangle = \left\langle T_{\gamma_1} f_i, T_{\gamma_2} f_j \right\rangle = \begin{cases} 1 & \text{if } \gamma_1 = \gamma_2 \quad \text{and} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

Thus if $i \neq j$,

$$\left\langle \boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi), \mathrm{e}^{\imath(\gamma_1-\gamma_2)\cdot\xi} \right\rangle = 0.$$

Since this is true for any choice of γ_1, γ_2, this means that

$$\left\langle \boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi), \mathrm{e}^{\imath\gamma\cdot\xi} \right\rangle = 0 \quad \text{for all } \gamma \in \mathbb{Z}^d.$$

In other words, all the Fourier coefficients of $\boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi)$ are zero. Now we can use the same technique as in part (i) to we show that $\boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi)$ converges to its Fourier series: there exists an L^2 function G whose Fourier coefficients are the Fourier coefficients of $\boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi)$. Then since G is the a.e. limit of its Fourier series, $G = 0$ a.e., so G is in $L^1 \cap L^2$. Now we verify that $\boldsymbol{\mu}_i(\xi) \cdot \boldsymbol{\mu}_j(\xi) \in L^1[0, 2\pi]^d$. We have

$$1 = \left\langle T_\gamma f_j, T_\gamma f_j \right\rangle = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \|\boldsymbol{\mu}_j(\xi)\|^2 \, \mathrm{d}m(\xi),$$

whence $\|\boldsymbol{\mu}_j(\xi)\| \in L^2[0, 2\pi]^d$ for each j. Now by Cauchy-Schwarz, we have

$$\left|\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j\right| \leq \|\boldsymbol{\mu}_i\| \cdot \|\boldsymbol{\mu}_j\|,$$

whence by Hölder's inequality, $\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j \in L^1[0, 2\pi]^d$. Now since

$$\widehat{\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j}(\gamma) = \widehat{G}(\gamma) \quad \text{for all } \gamma \in \mathbb{Z}^d,$$

and G and $\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j$ are both in L^1, Corollary 8.27 in Folland gives us that these two functions are equal a.e., so that $\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j$ is in L^2, and hence is equal to the sum of its Fourier series—zero. In the same way, we show that $\|\boldsymbol{\mu}_j\| = 1$ for each $j = 0, \dots, s$.

(iii) Suppose that the matrix (22) is unitary for a.e. $\xi \in L^2[0, 2\pi]^d$. Then the family $\{T_\gamma f_j\}_{\gamma\in\mathbb{Z}^d; j=0,\dots,q-1}$ is orthonormal, by part (ii). Suppose that this family does not form an orthonormal basis—then there must exist some $f_q \in L^2[0, 2\pi]^d$ such that

$$f_q \perp \{T_\gamma f_j\}_{\gamma\in\mathbb{Z}^d; j=0,\dots,q-1}.$$

Then by part (ii), the vectors $\{\boldsymbol{\mu}_0(\xi), \dots, \boldsymbol{\mu}_q(\xi)\}$ are orthogonal in $\mathbb{C}^q$ for a.e. ξ, which is impossible, since we cannot have $q + 1$ orthogonal vectors in $\mathbb{C}^q$.

Conversely, if the family $\{T_\gamma f_j\}_{\gamma\in\mathbb{Z}^d;j=0,\ldots,q-1}$ is orthonormal; then by part (ii), the corresponding vectors $\{\boldsymbol{\mu}_0(\xi),\ldots,\boldsymbol{\mu}_{q-1}(\xi)\}$ are orthonormal in $\mathbb{C}^q$ for a.e. ξ, and hence the matrix (22) is unitary for a.e. ξ.

(iv) This part amounts to finding a unitary matrix whose entries are measurable functions, and whose first row is given. The unitary property will ensure that the norm of each row is 1, so that the entries are bounded, and hence in $L^2[0,2\pi]^d$. The entries of the rows to be determined will be defined initially on $[0,2\pi]^d$; they can then be extended periodically to $\mathbb{R}^d$ in the natural way.

First, for each $i=0,\ldots,q-1$, define

$$E_i := \left\{\xi\in[0,2\pi]^d : \mu^i_\Phi(\xi)\neq 0\right\}.$$

Now define $F_0 := E_0$, and for $i=1,\ldots,q-1$ define

$$F_i := E_i \setminus \bigcup_{j=0}^{i-1} E_j.$$

The F_i partition $[0,2\pi]^d$ into q disjoint, measurable subsets. Consider the constant vector functions defined to be the standard basis vectors

$$\mathbf{e}_i(\xi) = \begin{bmatrix}\mathbf{0}(\xi) & \cdots & \mathbf{1}(\xi) & \cdots & \mathbf{0}(\xi)\end{bmatrix}, \tag{26}$$

with the constant function $\mathbf{1}(\xi)$ in the ith position, and the constant functions $\mathbf{0}(\xi)$ in all other positions. Note that these row vectors are linearly independent. We construct the matrix function $\mathbf{U}(\xi)$ as follows: if $\xi\in F_i$, we fill the remaining rows, in any order, with the row vectors $\mathbf{e}_j$, $j\neq i$. The resulting matrix must have linearly independent rows, since the first row is linearly independent of the remaining linearly independent rows: the first row being nonzero in the ith position, the other rows having 0 in the ith position. We then apply the Gram-Schmidt process to this matrix. The Gram-Schmidt process will produce rows whose entries are measurable functions of ξ. Indeed, if we have q linearly independent vectors $\mathbf{v}_1,\ldots,\mathbf{v}_q$, then the Gram-Schmidt process produces q orthonormal vectors $\mathbf{u}_1,\ldots,\mathbf{u}_q$ according to the following formal determinant formula:

$$\mathbf{u}_j = \frac{1}{\sqrt{D_{j-1}D_j}} \begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_1 & \cdots & \mathbf{v}_j \cdot \mathbf{v}_1 \\ \mathbf{v}_1 \cdot \mathbf{v}_2 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_j \cdot \mathbf{v}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_1 \cdot \mathbf{v}_{j-1} & \mathbf{v}_2 \cdot \mathbf{v}_{j-1} & \cdots & \mathbf{v}_j \cdot \mathbf{v}_{j-1} \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_j \end{vmatrix}, \tag{27}$$

where $D_0 = 1$ and, for $j \geq 1$, D_j is the Gram determinant

$$D_j = \begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_1 & \cdots & \mathbf{v}_j \cdot \mathbf{v}_1 \\ \mathbf{v}_1 \cdot \mathbf{v}_2 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_j \cdot \mathbf{v}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_j \cdot \mathbf{v}_1 & \mathbf{v}_j \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_j \cdot \mathbf{v}_j \end{vmatrix}. \tag{28}$$

The operations in calculating these determinants preserve measurability; this is a way of seeing that the Gram-Schmidt process does indeed preserve the measurability of the entries. Thus, for $\xi \in F_i$, we obtain a unitary matrix $\mathbf{W}_i(\xi)$, whose entries are measurable functions of ξ, and whose first row is

$$\boldsymbol{\mu}_\Phi(\xi) = \begin{bmatrix} \mu_\Phi^0(\xi) & \cdots & \mu_\Phi^1(\xi) & \cdots & \mu_\Phi^{q-1}(\xi) \end{bmatrix}. \tag{29}$$

Finally, we obtain the desired unitary matrix $\mathbf{U}(\xi)$ by taking

$$\mathbf{U}(\xi) = \sum_{j=0}^{q-1} \mathbf{W}_j(\xi)\chi_{F_j}(\xi),$$

where $\chi_{F_j}(\xi)$ denotes the indicator function on the measurable subset F_j. Defining $\mu_j^r(\xi)$, for $1 \leq j \leq q-1$ and $0 \leq r \leq q-1$, to be the jr-th entry $\mathbf{U}(\xi)_{jr}$ of the matrix $\mathbf{U}(\xi)$, we obtain equation (23). We thereby obtain—under the correspondence (17)—a collection $\{\Phi, \Psi_1, \ldots, \Psi_{q-1}\}$ of elements whose $\mathbb{Z}^d$-translates form an orthonormal basis for V_1, as per part (iii) of this theorem. $\square$

Corollary 1.15. Given a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ associated with the dilation matrix A, there exists a multiwavelet $\{\Psi_1, \ldots, \Psi_{q-1}\}$ consisting of $q-1$ elements, where $q = |\det A|$.

Proof. For each $j \in \mathbb{Z}$, let W_j denote the orthogonal difference $V_{j+1} \ominus V_j$; i.e.,

$$W_j = \{x \in V_{j+1} : x \perp V_j\}.$$

Then W_j is a closed subspace of $L^2(\mathbb{R}^d)$, for each j. Since $\bigcup_{j \in \mathbb{Z}} W_j = \bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$, we have $L^2(\mathbb{R}^d) = \bigvee_{j \in \mathbb{Z}} W_j$; and since the W_j are mutually orthogonal, this implies that $L^2(\mathbb{R}^d) = \bigoplus_{j \in \mathbb{Z}} W_j$.

Since $V_j = D_A^j(V_0)$ for each $j \in \mathbb{Z}$, where D_A is a unitary operator, it follows that $W_j = D_A^j W_0$. So if we can find a finite collection $\{\Psi_1, \dots, \Psi_{q-1}\}$ of elements whose $\mathbb{Z}^d$-translates form an orthonormal basis for W_0, we will have a multiwavelet. But from the preceding theorem, we know that there exist $\Psi_1, \dots, \Psi_{q-1} \in V_1$ such that the $\mathbb{Z}^d$-translates of elements in the collection

$$\{\Phi, \Psi_1, \dots, \Psi_{q-1}\}$$

form an orthonormal basis for V_1; and since $\{T_\gamma \Phi\}_{\gamma \in \mathbb{Z}^d}$ is an orthonormal basis for V_0, it follows that $\{\Psi_1, \dots, \Psi_{q-1}\}$ is the desired multiwavelet. $\square$

We remark that since there are many different ways of building the unitary matrix (23), the construction of a multiwavelet from a given MRA is highly nonunique.

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