MATH 2120 – Bonus questions

The first person to provide a fully justified answer (as determined by me) to all parts of any problem will have 5 points added to their midterm. There are no limits on how many each person may solve. Please ask for clarifications if required. These questions will remain open until they have been solved.

1. **[solved]** This question shows that we can still obtain useful properties of a solution even though we are unable to find an explicit expression for it.

   (a) Find an implicit solution to the differential equation

   $$\frac{dy}{dx} = \frac{1}{x+y}\,, \qquad y(x_0) = y_0\,.$$

   [hint: solve for $x = x(y)$]

   (b) Find the inequality relation (i.e., find $f(x_0)$)

   $$y_0 > f(x_0) \tag{1}$$

   between x_0 and y_0 for which the solution $y(x)$ has a vertical tangent at some point (x^*, y^*). Calculate x^* and y^* in terms of x_0 and y_0.

   (c) The presence of a vertical tangent (or "saddle node") indicates that the solution $y(x)$ only exists when $x < x^*$ or $x > x^*$. Show that it is the latter.

   (d) When the inequality (1) is satisfied (i.e., there is a saddle node), find the range of y_0 for which $y \to +\infty$ as $x \to \infty$, and the range of y_0 for which $y \to -\infty$ as $x \to \infty$. In both cases, determine the behavior of $y(x)$ as $x \to \infty$.

   (e) When the inequality (1) is reversed (i.e., there is no saddle node), the solution exists for all x. Determine the behaviors of $y(x)$ as $x \to -\infty$ and $+\infty$.

2. **[solved]** This question demonstrates how to solve (maybe nonlinear) second order differential equations of the form

   $$\frac{d^2u}{dx^2} = f(u)\,. \tag{2}$$

   For example, if $f(u) = 4u$, (2) would be a standard linear constant coefficient equation that we have seen in class. The procedure for this type of equation is to multiply the left and right-hand sides of (2) by du/dx, and then to do something. Part of your job is to figure out what that something is.

   (a) For $f(u) = u$ in (2), find the general solution using the procedure suggested above. The solution will of course have to agree with what we found in class.

(b) Repeat part (a) for $f(u) = -u$.

(c) For the equation

$$\frac{d^2u}{dx^2} + \gamma\frac{du}{dx} + u = 0\,, \tag{3}$$

find the appropriate substitution that transforms (3) into the equation considered in part (b). Here, $0 < \gamma < 2$ is a constant.

(d) Consider the nonlinear **boundary value problem**

$$\varepsilon^2\frac{d^2u}{dx^2} - mu + Au^2 = 0\,; \qquad u \to 0 \text{ as } |x| \to \infty\,. \tag{4}$$

Here, ε, m, and A are positive constants. Rescale (4) appropriately (as we did in other class examples) to show that it can be reduced to the form

$$\frac{d^2w}{dx^2} - w + w^2 = 0\,; \qquad w \to 0 \text{ as } |x| \to \infty\,. \tag{5}$$

(e) Without solving for u or w, deduce from (5) the asymptotic behavior of u as $x \to \pm\infty$.

(f) Solve (5) for w in terms of elementary functions (it should look very simple).

(g) Using the solution for w, write the solution of the unscaled equation (4) for u.

3. **[solved]** This question introduces you to asymptotic expansions and where it fails. An important thing to note in this question is that we already have the exact solution, so there is no point in trying to find an asymptotic solution. However, asymptotic techniques may be used in instances (e.g., nonlinear equations) where no exact solutions are available. The point is to test these techniques on equations for which we know the exact solution so that we can see where they work and where they fail.

(a) Solve the IVP

$$x'' + 2\,\varepsilon x' + x = 0\,; \qquad x(0) = 0\,, \quad x'(0) = 1\,. \tag{6}$$

(b) Assuming that $0 < \varepsilon \ll 1$, expand the solution of (6) to show that $x(t) \sim \sin t - \varepsilon t \sin t$. For what range of t should this expansion agree reasonably well with the exact solution?

(c) Assume that $x(t)$ has the asymptotic expansion $x \sim x_0(t) + x_1(t)$. Substitute this expansion into (6) and collect powers of ε to formulate IVP's for $x_0(t)$ and $x_1(t)$. Solve for $x_0(t)$ and $x_1(t)$. Your solution should agree with the expansion found in part (b).

(d) Use a plotting tool to plot the exact solution and the two-term expansion as functions of t. Repeat for a range of ε between, say, 1×10^{-1} and 1×10^{-5}. Observe in each case the approximate value of $t = t_{err}(\varepsilon)$ around which the solutions begin to diverge from each other.

(e) Plot $\log(t_{err}(\varepsilon))$ versus $\log(\varepsilon)$, and find the slope. Show that this is consistent with your prediction in part (b).

A "fix" that produces an asymptotic solution valid for longer times is available by way of a multiple scales analysis, which is beyond our scope for now. Note that, in the exact solution for $x(t)$, there are two time-scales: an $\mathcal{O}(1)$ time-scale over which the oscillations occur, and an $\mathcal{O}(\varepsilon)$ time-scale over which the slow amplitude decay occurs. The multiple scales analysis builds this fact into the solution. In fact, by accounting for enough time-scale corrections, one can construct an asymptotic solution that remains valid for arbitrarily large times.

4. **[unsolved]** This question motivates a more meaningful derivation of the formula for variation of parameters. The key is to convert a nonhomogeneous equation into a series of homogeneous equations with augmented initial conditions, which we can solve. Let us consider the equation

$$L(y) \equiv y'' + p(t)y' + q(t)y = f(t)\,; \qquad y(0) = \alpha\,, \quad y'(0) = \beta\,, \tag{7}$$

where two linearly independent homogeneous solutions are $y_1(t)$ and $y_2(t)$. Assume that $p(t)$ and $q(t)$ are continuous and differentiable everywhere. The solution to (7) may be broken down as

$$y = y_h + y_p\,, \tag{8}$$

where $L(y_h) = 0$, $y_h(0) = \alpha$, $y_h'(0) = \beta$, and $L(y_p) = f(t)$, $y_p(0) = y_p'(0) = 0$. To calculate y_p, let us first assume that $f(t) \equiv 0$ for all $t < 0$. Then we discretize $f(t)$ as

$$f(t) = \lim_{\Delta t \to 0} \sum_k f(t_k) \frac{b(t-t_k)}{\Delta t} \Delta t\,; \quad t_k \equiv k\Delta t\,. \tag{9a}$$

In (9a), the function $b(t-t_k)$ is defined as

$$b(t-t_k) = \begin{cases} 1\,, & t_k \le t \le t_{k+1}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{9b}$$

In particular, $b(t-t_k)$ has width Δt and height 1. Note the analogy of (9) to the continuous version

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\, \delta(t-\tau)\, d\tau\,,$$

with the Dirac delta function formally defined as

$$\delta(\tau - t) = \begin{cases} \infty\,, & \tau = t\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{10}$$

Notice that if we substitute (9) into (7), we may apply the principle of linearity. All we have to do then is to solve the equation

$$y''_{pk} + p(t)y'_{pk} + q(t)y_{pk} = f(t_k)b(t - t_k)\,; \qquad y_{pk}(0) = 0\,, \quad y'_{pk}(0) = 0\,, \tag{11}$$

and express the particular solution y_p as

$$y_p = \lim_{\Delta t \to 0} \sum_k y_{pk}\,. \tag{12}$$

To calculate y_{pk}, we may use the fact that $b(t - t_k)$ has very small duration to convert (11) into a homogeneous equation with nonhomogeneous initial conditions. In particular, integrate (11) from $t : t_k \to t_{k+1}$ to calculate the "jump in velocity" that occurs due to the $f(t_k)b(t - t_k)$ term. You should end up with (make a simple argument as to why the change in position over this interval must be $\mathcal{O}(\Delta t^2)$ and may therefore be ignored),

$$L(y_{pk}) = 0\,; \qquad y_{pk}(t_{k+1}) = 0\,, \quad y'_{pk}(t_{k+1}) = f(t_k)\Delta t\,. \tag{13}$$

Solve (13) and use (12) to obtain the formula for the variable coefficients $u_1(t)$ and $u_2(t)$ we derived in class using a completely different method.