Bonus problems

1 $\dfrac{dy}{dx} = \dfrac{1}{x+y}$, $y(x_0) = x_0$

let $y = h(x) \Rightarrow \dfrac{dy}{dx} = h'(x)$

diff. w.r.t ~~x~~ y : $1 = \cancel{\dfrac{dy}{dy}}\, h'(x) \dfrac{dx}{dy}$

so

$$\frac{dx}{dy} = \frac{1}{h'(x)} = \frac{1}{dy/dx}$$

$\Rightarrow \dfrac{dx}{dy} = x + y$ $\leftarrow$ liner first order eqn

solve by integrating factor...

$$x(y) = -1 - y + ce^{y}$$

~~x(y_0)=~~ $y(x_0) = y_0$:

$$x_0 = -1 - y_0 + ce^{y_0}$$

$$x_0 + 1 + y_0 = c\, e^{y_0}$$

$$c = e^{-y_0}(1 + x_0 + y_0)$$

so

$$\boxed{x(y) = -1 - y + e^{y - y_0}(1 + x_0 + y_0)}$$

b) vertical tangent in $y(x)$ $\Rightarrow$ slope zero in $x(y)$

$$\frac{dx}{dy} = -1 + e^{y - y_0}(1 + x_0 + y_0) = 0$$

$$1 + x_0 + y_0 = \underbrace{e^{y_0 - y}}_{\text{always positive}}$$

so we require

$$1 + x_0 + y_0 > 0 \Rightarrow \boxed{y_0 > -1 - x_0}$$

B3

back to $1+x_0+y_0 = e^{y_0-y^*}$

$$\log[1+x_0+y_0] = y_0 - y^*$$

$$\boxed{y^* = y_0 - \log(1+x_0+y_0)}$$

since $\frac{dx}{dy} = x+y = 0$, $\boxed{x^* = -y^*}$

c) let $x = x^* + \gamma \qquad |\gamma| \ll 1$

$y = y^* + \varepsilon \qquad |\varepsilon| \ll 1$

$$x^* + \gamma = -1 - y^* - \varepsilon + e^{y^* - y_0 + \varepsilon}(1+x_0+y_0)$$

since $x^* = -y^*$,

$$\gamma = -1 - \varepsilon + e^{y^*-y_0}(1+x_0+y_0)e^{\varepsilon}$$

now with $1+x_0+y_0 = e^{y_0 - y^*}$,

we have

$$\gamma = -1 - \varepsilon + e^{\varepsilon}$$

$\cancel{\gamma \approx}$

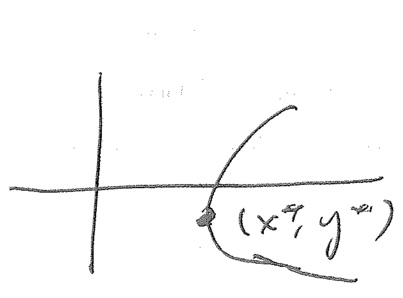
$$\gamma = -1 - \varepsilon + 1 + \varepsilon + \frac{\varepsilon^2}{2}$$

so

$$\gamma = \frac{\varepsilon^2}{2} > 0$$

since we have only 1 saddle, and we know it bends toward positive x ($\gamma > 0$), we conclude that y only exists for $x > x^*$.

d) we must have that $x(y)$ looks like



so for $y_0 > y^*$, $y \to +\infty$ as $x \to +\infty$.

and for $y_0 < y^*$, $y \to -\infty$ as $x \to +\infty$.

for $y_0 > y^*$, and $y \to +\infty$, we have that $|e^{y-y_0}| \gg |y|$. So the dominant balance is between x and e^{y-y_0}

$$\Rightarrow \quad x \sim e^{y-y_0}(1+x_0+y_0)$$

$$\frac{x}{1+x_0+y_0} \approx e^{y-y_0}$$

$$\log\left(\frac{x}{1+x_0+y_0}\right) \sim y - y_0$$

so

$$\boxed{y \sim y_0 + \log\left(\frac{x}{1+x_0+y_0}\right)}$$

as $x \to +\infty$.

for $y_0 < y^*$ with $y \to -\infty$ as $x \to +\infty$,

$|e^{y-y_0}| << |y|$ so dominant balance

is

$x \sim -1 - y$ so $\boxed{y \sim -1 - x \text{ as } x \to \infty}$

e) now with $1 + x_0 + y_0 < 0$, and $x \to -\infty$,

if the dominant balance is between

x and y, then ~~$y \to \ldots x$~~ $y \sim -x - 1$

as $x \to -\infty$. But then $|e^{y-y_0}| >> |y|$, so

we must have

$$x \sim -e^{y-y_0} \, |1 + x_0 + y_0|$$

$$\frac{+|x|}{|1 + x_0 + y_0|} \sim e^{y-y_0}$$

then

$$\boxed{y \sim y_0 + \log\left|\frac{x}{1+x_0+y_0}\right| \quad \text{as} \quad x \to -\infty}$$

as $x \to +\infty$, we must have instead

$$x \sim -1-y$$

so $\boxed{y \sim -1-x \quad \text{as} \quad x \to +\infty}$

② a) $u'' = u \qquad u = u(x)$

$$u'u'' = u'u$$

$$\frac{1}{2}\frac{d}{dx}(u')^2 = \frac{1}{2}\frac{d}{dx}u^2$$

$$(u')^2 = u^2 + c$$

$$u' = \sqrt{u^2+c} \quad \Rightarrow \quad \cancel{\frac{dy}{dt}} \; \frac{du}{dx} = \sqrt{u^2+c}$$

$$\int \frac{du}{\sqrt{u^2+c}} = \int dx$$

to integrate LHS, let $u = \sqrt{c}\sinh\theta$

$du = \sqrt{c}\cosh\theta\, d\theta$

$$\int \frac{du}{\sqrt{u^2+c}} = \int \frac{\sqrt{c}\cosh\theta}{\sqrt{c\sinh^2\theta + c}}\, d\theta$$

$$= \int \frac{\sqrt{c}\cosh\theta}{\sqrt{c}\sqrt{\sinh^2\theta+1}}\, d\theta = \int \frac{\sqrt{c}\cosh\theta}{\sqrt{c}\cosh\theta}\, d\theta$$

$$\Rightarrow \int d\theta = \int dx$$

so $\theta = x + k$ for constant k.

then we have

$$u = \sqrt{c}\sinh(x+k)$$

now we use

$$\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$$

so we have

$$u = \sqrt{c} \left[\sinh x \cosh k + \sinh k \cosh x \right]$$

$$= \sqrt{c} \cosh k \sinh x + \sqrt{c} \cosh x \sinh k$$

let $\sqrt{c} \cosh k = A$, $\sqrt{c} \sinh k = B$

then

$$u = A \sinh x + B \cosh x$$

as required.

b) $u'' = -u$

following the same procedure, we have

$$\frac{du}{dx} = \sqrt{c - u^2}$$

$$\int \frac{du}{\sqrt{c-u^2}} = \int dx$$

let $u = \sqrt{c}\sin\theta$

$du = \sqrt{c}\cos\theta\, d\theta$

so

$$\int \frac{du}{\sqrt{c-u^2}} = \int \frac{\sqrt{c}\cos\theta\, d\theta}{\sqrt{c - c\sin^2\theta}}$$

$$= \int d\theta$$

$$\Rightarrow \int d\theta = \int dx$$

$\theta = x + k.$

then $u = \sqrt{c}\sin(x+k) = \sqrt{c}\left[\sin x\cos k + \sin k\cos x\right]$

$= \sqrt{c}\cos k\sin x + \sqrt{c}\sin k\cos x$

then with

$$A = \sqrt{c}\cos k, \qquad B = \sqrt{c}\sin k,$$

we have

$$u = A\sin x + B\cos x \qquad \text{as required.}$$

c) $u'' + \gamma u' + u = 0$

let $u = e^{-\frac{\gamma}{2}x} v$

$$u' = -\frac{\gamma}{2} e^{-\frac{\gamma}{2}x} v + v' e^{-\frac{\gamma}{2}x}$$

$$u'' = \frac{\gamma^2}{4} e^{-\frac{\gamma}{2}x} v - \frac{2\gamma}{2} e^{-\frac{\gamma}{2}x} v' + v'' e^{-\frac{\gamma}{2}x}$$

so

$$v'' \cancel{- \frac{2\gamma}{2} v'} + \frac{\gamma^2}{4} v - \frac{\gamma^2}{2} v + \cancel{\gamma v'} + v$$

$$v'' + \left(1 - \frac{\gamma^2}{4}\right) v = 0.$$

let $\Gamma = 1 - \frac{\gamma^2}{4} > 0$

then

$$V'' + \Gamma V = 0$$

let $x = y/\sqrt{\Gamma}$ and $V(x) = w(y)$

then

$$\frac{dV}{dx} = \sqrt{\Gamma}\,\frac{dw}{dy} \Rightarrow \frac{d^2V}{dx^2} = \Gamma\,\frac{d^2w}{dy^2}$$

then

$$\frac{d^2w}{dy^2} + w = 0 \qquad \text{as desired.}$$

d) $\varepsilon^2 u'' - mu + Au^2 = 0$; $u \to 0$ as $|x| \to \infty$.

$$\frac{\varepsilon^2}{m} u'' - u + \frac{A}{m} u^2 = 0$$

let $x = \cancel{\frac{\sqrt{m}}{\varepsilon} y}\ \frac{\varepsilon}{\sqrt{m}}\, y$, $u(x) = v(y)$

then $\dfrac{d^2}{dx^2} \to \dfrac{m}{\varepsilon^2} \dfrac{d^2}{dy^2}$

so

$$v'' - v + \frac{A}{m} v^2 = 0.$$

now let $v = a w$

$$a w'' - a w + a^2 \frac{A}{m} w^2 = 0.$$

choose $a^2 \frac{A}{m} = a \Rightarrow a = \frac{m}{A}$

therefore

$$w'' - w + w^2 = 0 \quad , w \to 0 \text{ as } |y| \to \infty.$$

e) since $w \to 0$ as $|y| \to \infty$, we have

$w^2 << w$ as $|y| \to \infty$.

$$\Rightarrow w'' = w$$

so we have $w \sim c e^{-|y|}$ as $|y| \to \infty$.
or $u \sim \hat{c} e^{-\frac{\sqrt{m}}{\varepsilon}|x|}$ as $|x| \to \infty$

f) $w'' - w + w^2 = 0$

$$w'' = -w^2 + w$$

$$w' w'' = w' \left[w - w^2 \right].$$

$$\frac{1}{2} \frac{d}{dy} (w')^2 = \frac{d}{dy} \left(\frac{w^2}{2} - \frac{w^3}{3} \right)$$

$$\frac{(w')^2}{2} = \frac{w^2}{2} - \frac{w^3}{3} + c$$

so $w, w' \to 0$ as $|y| \to \infty$, $c = 0$.

$$(w')^2 = w^2 - \frac{2}{3} w^3 = w^2 \left(1 - \frac{2}{3} w\right).$$

$$w' = w \sqrt{1 - \frac{2}{3} w}.$$

$$\int \frac{dw}{w \sqrt{1 - \frac{2}{3} w}} = \int dy$$

let $w = \frac{3}{2} \operatorname{sech}^2 \theta$

$dw = -3 \operatorname{sech}^2 \theta \tanh \theta \, d\theta.$

$$\int \frac{dw}{w\sqrt{1-\frac{2}{3}w}} = \int \frac{-3\,\mathrm{sech}^2\theta\,\tanh\theta}{\frac{3}{2}\mathrm{sech}^2\theta\,\sqrt{1-\mathrm{sech}^2\theta}}\,d\theta$$

now

$$1-\mathrm{sech}^2\theta = \tanh^2\theta$$

$$= \int \frac{dw}{w\sqrt{1-\frac{2}{3}w}} = -\int 2\,d\theta$$

$$\Rightarrow \quad -2\theta = y + c$$

take $c = 0$ to center maximum at $y=0$.

then $\theta = -\frac{y}{2}$.

but $\mathrm{sech}^2\left(\frac{-y}{2}\right) = \mathrm{sech}^2(y/2)$

$$\Rightarrow \boxed{w = \frac{3}{2}\,\mathrm{sech}^2(y/2)}$$

g) now $v = \frac{m}{A} w = \frac{3m}{2A}\,\mathrm{sech}^2(y/2)$

and

$$x = \frac{\varepsilon}{\sqrt{m}}\, y \,,$$

$$\Rightarrow \quad \cancel{u(x) = \frac{3m}{2A}\,\mathrm{sech}^2\left(\frac{\varepsilon}{2\sqrt{m}}\, x\right)}$$

$$\boxed{u(x) = \frac{3m}{2A}\,\mathrm{sech}^2\left(\frac{\sqrt{m}}{2\varepsilon}\, x\right)}$$

(3) a) $x'' + 2\varepsilon x' + x = 0, \quad x(0) = 0, \quad x'(0) = 1$

C.E: $r^2 + 2\varepsilon r + 1 = 0$

$$r = \frac{-2\varepsilon \pm \sqrt{4\varepsilon^2 - 4}}{2} = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 1}}{1}$$

with $\varepsilon \ll 1$,

$$r = -\varepsilon \pm i\sqrt{1-\varepsilon^2}$$

$$x = e^{-\varepsilon t}\left[A\cos\omega t + B\sin\omega t\right], \quad \omega \equiv \sqrt{1-\varepsilon^2}$$

$x(0) = 0 \Rightarrow A = 0$

$x'(0) = \cancel{\varepsilon B}\; B\omega = 1 \Rightarrow B = \frac{1}{\omega}$

so

$$\boxed{x(t) = \frac{1}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon t} \sin\sqrt{1-\varepsilon^2}\, t}$$

b) suppose $\varepsilon t << 1$, then $e^{-\varepsilon t} \sim 1 - \varepsilon t + O(\varepsilon^2)$

$$(1-\varepsilon^2)^{-1/2} \sim 1 + \frac{1}{2}\varepsilon^2$$

$$(1-\varepsilon^2)^{1/2} \sim 1 - \frac{1}{2}\varepsilon^2$$

then $\sin\sqrt{1-\varepsilon^2}\, t \sim \sin\left(t - \frac{\varepsilon^2}{2} t\right) \sim \sin t + O(\varepsilon^2)$

$\Rightarrow x(t) \sim (1-\varepsilon t)\sin t + O(\varepsilon^2)$ for $\varepsilon t << 1$

the expansion $e^{-\varepsilon t} \sim 1 - \varepsilon t$ required $\varepsilon t << 1$

so the approximation holds only for

$$0 < t << \frac{1}{\varepsilon}.$$

c) assume $x(t) \sim x_0(t) + \varepsilon x_1(t)$

sub into ODE:

$$x_0'' + \varepsilon x_1'' + 2\varepsilon(x_0' + \varepsilon x_1') + x_0 + \varepsilon x_1 = 0$$

$x_0(0) + \varepsilon x_1(0) = 0$

$x_0'(0) + \varepsilon x_1'(0) = 1$

collect powers of ε:

$O(1)$: $x_0'' + x_0 = 0$, $\quad x_0(0) = 0$, $x_0'(0) = 1$

$\Rightarrow x_0(t) = \sin t$.

$O(\varepsilon)$: $x_1'' + 2x_0' + x_1 = 0$, $\quad x_1(0) = x_1'(0) = 0$.

$x_1'' + x_1 = -2x_0' = -2\cos t$

$x_1 = A\cos t + B\sin t - t\sin t$.

$x_1(0) = 0 \Rightarrow A = 0$

$x_1'(0) = B = 0 \Rightarrow B = 0$.

$x_1(t) = -t\sin t$

notice: εt better be "small" or else the assumption that $\varepsilon x_1(t) \ll x_0(t)$ would be ~~violation~~ violated.

$\Rightarrow \varepsilon t \ll 1$ for validity.

$\Rightarrow x(t) \sim \sin t - \varepsilon t \sin t + O(\varepsilon^2)$

(in agreement with part b)

d) $\varepsilon = 0.1$: $t_{err} \sim 10.52$

$\varepsilon = 0.01$: $t_{err} \sim 101.8$

$\varepsilon = 0.001$: $t_{err} \sim 1000.55$

$\varepsilon = 0.0001$: $t_{err} \sim 10001.24$

clearly, $t_{err} \sim \frac{1}{\varepsilon}$, consisent with our assertion that the asymptotic solution ~~is~~ is valid only when $t << \frac{1}{\varepsilon}$.

here, t_{err} is the first time at which the quantity

$$\cancel{\frac{|x_e(t) - x_a|}{e^{-\varepsilon t}}} \qquad \frac{|x_a(t) - x_\varepsilon(t)|}{e^{-\varepsilon t}}$$

exceeds unity (some measure of the relative error). Here, $x_a(t)$ is the asymptotic solution from part c), $x_e(t)$ is the exact solution from part a).

(4) $$y'' + p(t)y' + q(t)y = f(t), \quad y(0) = \alpha, \ y'(0) = \beta$$

let us assume we have two l.i. solutions of the homog. equation y_1, y_2.

let $y = y_h + y_p$, where

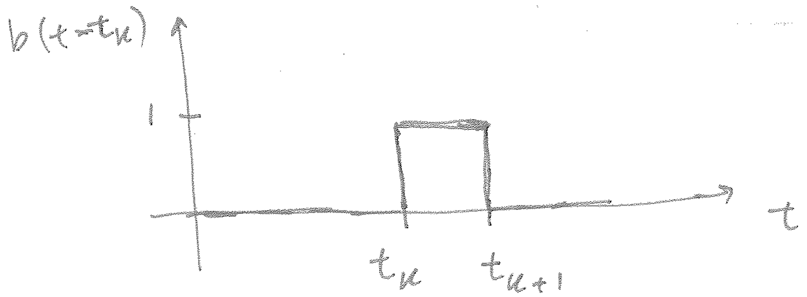
$$\mathcal{L}(y_h) = 0, \quad y_h(0) = \alpha, \quad y_h'(0) = \beta$$

$$\mathcal{L}(y_p) = f(t), \quad y_p(0) = y_p'(0) = 0 \qquad (*)$$

(add these together to see that indeed this satisfies original IVP)

write $f(t)$ as

$$f(t) = \lim_{\Delta t \to 0} \sum_k f(t_k) \frac{b(t - t_k)}{\Delta t} \Delta t \ ; \quad t_k \equiv k\Delta t$$



$$b(t - t_k) = \begin{cases} 1 & t_k < t < t_{k+1} \\ 0 & \text{else} \end{cases}$$

sub into $\circledast$:

$$L(y_p) = \sum_k f(t_k) \frac{b(t-t_k)}{\Delta t} \Delta t \quad ; \quad t_k \equiv k \Delta t$$

$$y_p(0) = y_p'(0) = 0$$

by linearity, we may express y_p as

$$y_p = \sum_k y_{pk}(t),$$

where

$$y_{pk}'' + p(t) y_{pk}' + q(t) y_{pk} = f(t_k) b(t-t_k)$$

$$y_{pk}(0) = y_{pk}'(0) = 0$$

notice that $L(y_{pk}) = 0$, $y_{pk}(0) = y_{pk}'(0) = 0$ for $0 < t < t_k$. $\Rightarrow$ $y_{pk} = 0$ for $0 \le t \le t_k$. A "kick" of duration Δt and strength ~~$f(t_k$~~ $f(t_k)\Delta t$ is applied at $t = t_k$. To calculate its effect, integrate ODE from $t = t_k \to t_{k+1}$ (interval of length Δt):

then we have

$$y_{pk}'(t_{k+1}) - \underset{\uparrow}{y_{pk}'(t_k)} + \int_{t_k}^{t_{k+1}} p(t)\, y_{pk}'\, dt + \int_{t_k}^{t_{k+1}} q(t)\, y_{pk}\, dt = f(t_k)\,\Delta t$$

0 since $y_{pk} = 0$ for $t \le t_k$

notice since $p(t)$ is finite and continuous, we must have

$$y_{pk}'(t_{k+1}) \gg \int_{t_k}^{t_{k+1}} p(t)\, y_{pk}'\, dt$$

since the interval of integration is $O(\Delta t)$.

$$\Rightarrow y_{pk}'(t_{k+1}) = O(\Delta t), \quad \int_{t_k}^{t_{k+1}} p(t)\, y_{pk}'(t)\, dt = O(\Delta t^2),$$

then since $y_p' = O(\Delta t)$, y_p must be $O(\Delta t^2)$ (since $y_p \approx \Delta t\, y_p'$) on the interval $t_k < t < t_{k+1}$

$\Rightarrow$ matching orders of Δt, we have

$$y_{pk}'(t_{k+1}) = f(t_k)\,\Delta t$$

$$y_{pk}(t_{k+1}) = O(\Delta t^2)$$

so we have for y_{pk}:

$$\mathcal{L}(y_{pk}) = 0 \,, \quad y_{pk}(t_{k+1}) = 0$$
$$y_{pk}'(t_{k+1}) = f(t_k)\,\Delta t$$

then

$$y_{pk}(t) = A_k y_1(t) + B_k y_2(t)$$

$$y_{pk}(t_{k+1}) = 0 \Rightarrow A_k y_1(t_{k+1}) + B_k y_2(t_{k+1}) = 0$$

$$y_{pk}'(t_{k+1}) \overset{f(t_k)\Delta t}{=} \Rightarrow A_k y_1'(t_{k+1}) + B_k y_2'(t_{k+1}) = f(t_k)\,\Delta t.$$

$$A_k = \frac{\begin{vmatrix} 0 & y_2(t_{k+1}) \\ f(t_k)\Delta t & y_2'(t_{k+1}) \end{vmatrix}}{\begin{vmatrix} y_1(t_{k+1}) & y_2(t_{k+1}) \\ y_1'(t_{k+1}) & y_2'(t_{k+1}) \end{vmatrix}} \,, \quad B_k = \frac{\begin{vmatrix} y_1(t_{k+1}) & 0 \\ y_1'(t_{k+1}) & f(t_k)\Delta t \end{vmatrix}}{\begin{vmatrix} y_1(t_{k+1}) & y_2(t_{k+1}) \\ y_1'(t_{k+1}) & y_2'(t_{k+1}) \end{vmatrix}}$$

$$A_k = \frac{-y_2(t_{k+1})\, f(t_k)\, \Delta t}{W_k}$$

$$B_k = \frac{y_1(t_{k+1})\, f(t_k)\, \Delta t}{W_k}$$

$$W_k \equiv \begin{vmatrix} y_1(t_{k+1}) & y_2(t_{k+1}) \\ y_1'(t_{k+1}) & y_2'(t_{k+1}) \end{vmatrix}$$

lastly, we have

$$y = \lim_{\Delta t \to 0} \sum_k y_{pk} = \lim_{\Delta t \to 0} \sum_k A_k y_1(t) + B_k y_2(t)$$

$$= y_1(t) \lim_{\Delta t \to 0} \sum_k A_k + y_2(t) \lim_{\Delta t \to 0} \sum_k B_k$$

$$= y_1(t) \lim_{\Delta t \to 0} \sum_k - \frac{y_2(t_{k+1}) f(t_k)}{W_k} \Delta t$$

$$+ y_2(t) \lim_{\Delta t \to 0} \sum_k \frac{y_1(t_{k+1}) f(t_k)}{W_k} \Delta t$$

in the limit $\Delta t \to 0$, $t_{k+1} \to t_k$

then we have

$$y(t) = c_1(t)\, y_1(t) + c_2(t)\, y_2(t),$$

$$c_1(t) \equiv -\int \frac{y_2(t) f(t)}{W(y_1, y_2; t)}\, dt \quad , \quad c_2 \equiv \int \frac{y_1(t) f(t)}{W(y_1, y_2; t)}\, dt$$