Review

(R1)

MUC

for equations of the form

$$ay'' + by' + cy = f(t) \quad (*).$$

with

$$f(t) = e^{\alpha t} P_n(t) \cos\beta t + e^{\alpha t} Q_n(t) \sin\beta t$$

$P_n(t)$, $Q_n(t)$ are order n polynomials

(e.g., $P_0 + t^n$)

(1) solve first the homog. problem

$$ay'' + by' + cy = 0$$

$$\Rightarrow y_1, y_2$$

(2) for particular soln, make guess

$$y_p = t^s e^{\alpha t}\Big[(a_0 + \dots a_n t^n) \cos\beta t + (b_0 + \dots + b_n t^n) \sin\beta t \Big].$$

R2

s is the smallest integer s.t. no term in y_p is a multiple of y_1 or y_2.

(3) sub y_p into (A) to calculate $a_0 \cdots a_n$, $b_0 \cdots b_n$ by equating coefficients of $t^k e^{\alpha t} \cos\beta t$, $t^k e^{\alpha} \sin\beta t$, $k = 0, \dots n$

(4) general soln is $y = c_1 y_1 + c_2 y_2 + y_p$.

(5) c_1, c_2 from IC's.

<u>Ex</u>

$$y'' - y' - 2y = (4 + t^2 + t^4) e^{3t}.$$

$$y'' - y' - 2y = 0 \Rightarrow y_1 = e^{2t}, \; y_2 = e^{-t}.$$

$$y_p = (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4) e^{3t}$$

Ex

$$y'' + y = 2t e^{3t} \sin t.$$

$$y'' + y = 0 \Rightarrow y_1 = \cos t, \quad y_2 = \sin t.$$

$$y_p = e^{3t}\left[(a_0 + a_1 t)\cos t + (b_0 + b_1 t)\sin t \right]$$

$(s=0)$

Ex (~~quiz~~ midterm).

$$y'' - (1+i)y' + iy = \cancel{t e^{i}}\ t \sin t$$

$$= t\frac{e^{it}}{2i} - t\frac{e^{-it}}{2i}$$

$$y'' - (1+i)y' + iy = 0.$$

$$r^2 - (1+i)r + i = 0.$$

$$(r-i)(r-1) = 0 \Rightarrow r = i, 1$$

$$y_1 = e^{it}, \quad y_2 = e^{t}.$$

$$y_p = (a_0 + a_1 t)e^{-it} + t(b_0 + b_1 t)e^{it}$$

R4

<u>Forced oscillations</u> $\quad mx'' + kx = F_0 \cos\omega t.$

beats, resonance, practical resonce

undamped $\qquad\qquad$ damped.

ω near $\sqrt{\frac{k}{m}} = \omega_0 \qquad \omega = \omega_0$

<u>beats</u>.

$$mx'' + kx = F_0 \cos\omega t, \qquad x(0) = x'(0) = 0.$$

$$x'' + \frac{k}{m}x = 0 \quad \Rightarrow \quad r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0.$$

$$x_h = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t.$$

$$x_p = a\cos\omega t + \cancel{b \sin\omega t}.$$

$$x_p'' = -\omega^2 a \cos\omega t.$$

$$-\omega^2 a m \cos\omega t + ka\cos\omega t = F_0 \cos\omega t$$

$$\Rightarrow \quad a = \frac{F_0/m}{\omega_0^2 - \omega^2}, \qquad \omega_0^2 = \frac{k}{m}.$$

so general soln is.

$$x = c_1 \cos w_0 t + c_2 \sin w_0 t + \frac{F_0/m}{w_0^2 - w^2} \cos wt.$$

$$x(0) = 0 \Rightarrow c_1 = -\frac{F_0/m}{w_0^2 - w^2}.$$

$$x'(0) = 0 \Rightarrow c_2 = 0.$$

$$x(t) = \frac{F_0/m}{w_0^2 - w^2} \left[\cos wt - \cos w_0 t \right].$$

recall

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$-\quad \cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B.$$

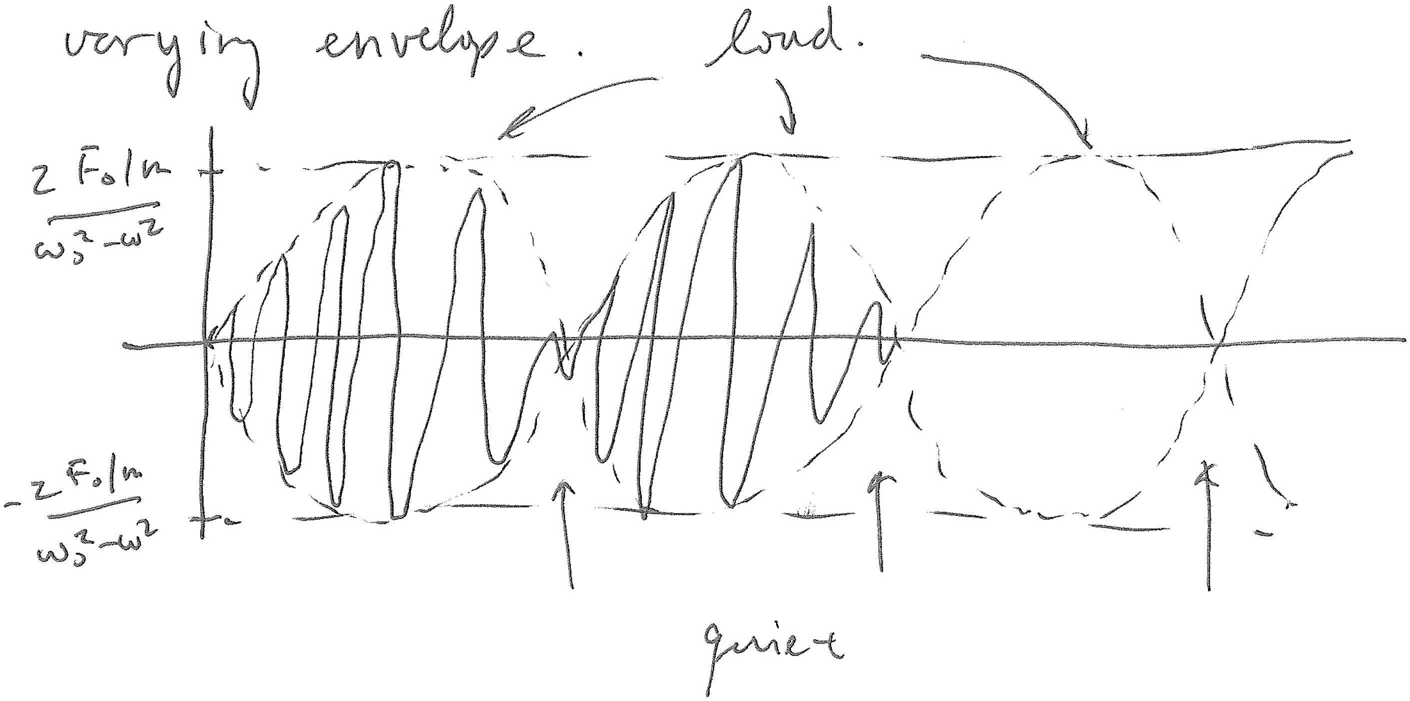
$$\left.\begin{aligned} A - B &= wt \\ A + B &= w_0 t \end{aligned}\right| \Rightarrow \begin{aligned} A &= \frac{w_0 + w}{2} t. \\ B &= \frac{w_0 - w}{2} t. \end{aligned}$$

$$x(t) = \boxed{\frac{F_0/m}{w_0^2 - w^2} 2 \cdot \sin\left(\frac{w_0 - w}{2} t\right)} \sin\left(\frac{w_0 + w}{2} t\right).$$

R6

if $\omega \simeq \omega_0$ $\quad |\omega - \omega_0| << 1$

so $\sin\left(\frac{\omega_0 - \omega}{2} t\right)$ plays role of slowly varying envelope.



<u>resonance</u> if $\omega = \omega_0$, (forcing freq. = natural freq) must augment guess for y_p:

$$x'' + \omega_0^2 x = \frac{F_0}{m} \cos\omega_0 t. \qquad \omega_0^2 = \frac{k}{m}$$

$x_1 = \cos\omega_0 t$, $\quad x_2 = \sin\omega_0 t$

$$x_p = t\left[A\cos\omega_0 t + B\sin\omega_0 t\right]$$

(use $(fg)'' = f''g + 2f'g' + fg''$

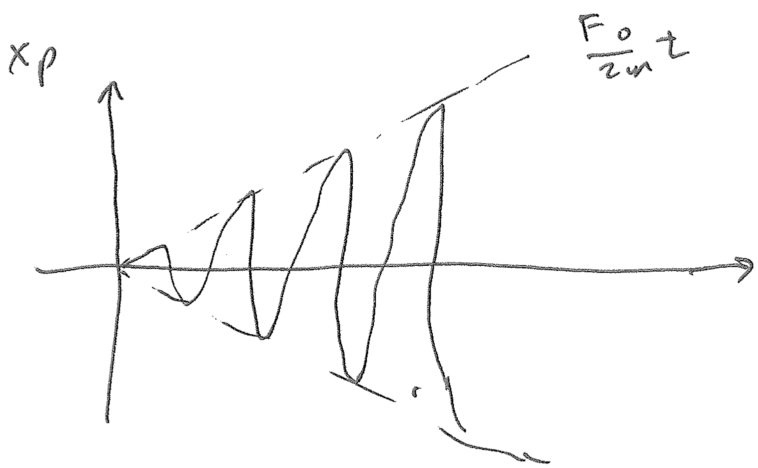
$g = t, \quad f = A\cos\omega_0 t + B\sin\omega_0 t.$

$$x_p'' = t\left[-\omega_0^2\right]\left[A\cos\omega_0 t + B\sin\omega_0 t\right]$$
$$+ 2\left[-\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t\right]$$
$$+ 0.$$

$$\Rightarrow x_p'' + \omega_0^2 x_p = 2\left[-\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t\right].$$
$$= \frac{F_0}{m}\cos\omega_0 t.$$

$$A = 0, \qquad B = \frac{F_0}{2m\omega_0}.$$

$$x(t) = \underbrace{c_1\cos\omega_0 t + c_2\sin\omega_0 t}_{\text{bounded for all } t} + \underbrace{\boxed{\frac{F_0}{2m} t}\sin\omega_0 t}_{\text{unbounded as } t\to\infty}.$$



practical resonance

$$mx'' + cx' + kx = F_0 \cos\omega t.$$

$m, c, k > 0.$

C.E: $mr^2 + cr + k = 0.$

$$r_\pm = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$Re(r_\pm) < 0.$, so $x_{1,2} = e^{r_\pm t} \to 0$

as $t \to \infty$. leaving only x_p.

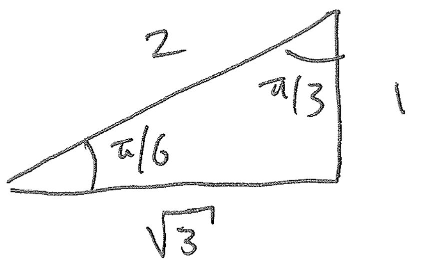
$$x_p = A\cos\omega t + B\sin\omega t$$

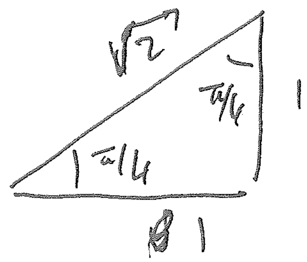
$$A = \frac{(k - m\omega^2) F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}.$$

$$x_p = C \cos(\omega t - \alpha)$$

$$C = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

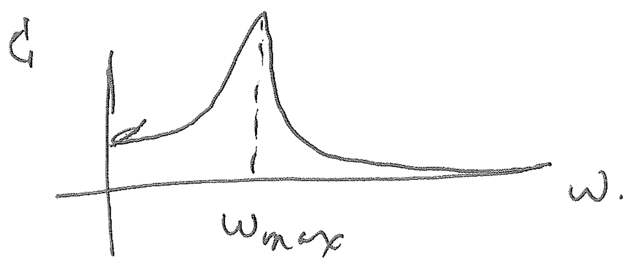
$$\alpha = \begin{cases} \tan^{-1}(B/A) & \text{if } A > 0 \\ \tan^{-1}(B/A) + \pi & \text{if } A < 0 \end{cases}$$





practical resonance occurs for ω when $G(\omega)$ is maximized.

$$\frac{dG}{d\omega} = 0 \Rightarrow \omega_{max} = \_\_\_\_\_\_$$



<u>Laplace transform of convolution</u>.

let $\mathcal{L}(f(t)) = F(s)$, $\mathcal{L}(g(t)) = G(s)$.

then

for $h(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t g(\tau) f(t-\tau) d\tau$

$$\mathcal{L}(h(t)) = F(s)\,G(s)$$

application to Volterra equation:

Ex (Quiz)

$$\phi(t) + \int_0^t (t-\tau)\,\phi(\tau)\,d\tau = \sin 2t.$$

$$\mathcal{L}(\phi(t)) = \Phi(s)$$

for convolution term, $f(t) = t$, $g(t) = \phi(t)$.

$$\mathcal{L}\left(\int_0^t \cdots\right) = \frac{1}{s^2}\,\Phi(s)$$

$$\Phi(s)\left[1 + \frac{1}{s^2}\right] = \frac{2}{s^2+4}$$

$$\Phi(s)\left[\frac{s^2+1}{s^2}\right] = \frac{2}{s^2+4}$$

$$\Phi(s) = \frac{2s^2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

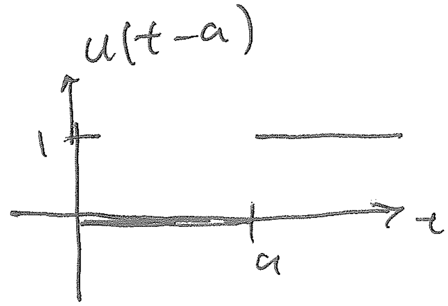
$$\Rightarrow (As+B)(s^2+4) + (Cs+D)(s^2+1) = 2s^2.$$

$s = 2i \Rightarrow C, D$

$s = i \Rightarrow A, B.$ etc...

step functions

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$



$$\mathcal{L}\left(u(t-a) f(t-a) \right) = e^{-as} F(s). \quad \left[\mathcal{L}(\delta(t-a)) = e^{-as} \right].$$

$$u(t-a) f(t-a) = \mathcal{L}^{-1}\left(e^{-as} F(s) \right)$$

Ex invert

$$e^{-as} \frac{1}{s^2 + 6s + 10}.$$

use theorem, indentify $F(s) = \dfrac{1}{s^2+6s+10}.$

$$= \frac{1}{s^2+6s+9+1} = \frac{1}{(s+3)^2+1^2}$$

$$\mathcal{L}^{-1}(F(s)) = f(t) = e^{-3t} \sin t.$$

$$\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = u(t-a)\, f(t-a)$$

$$= u(t-a)\, e^{-3(t-a)} \sin(t-a).$$

comes up when δ - kicks are part of the forcing term

Ex (Quiz 6 #1)

$$x'' + 4x' + 13x = \sum_{n=0}^{\infty} (-1)^n \delta\left(t - n\frac{\pi}{3}\right)$$

$$x(0) = x'(0) = 0.$$

$\mathcal{L}(\Sigma)$: do term by term.

$$\mathcal{L}\left(\delta(t - n\pi/3)\right) = \int_0^\infty e^{-st}\, \delta\left(t - n\frac{\pi}{3}\right) dt$$

$$= e^{-sn\pi/3}$$

$$(s^2 + 4s + 13)\, X(s) = \sum_{n=0}^{\infty} (-1)^n e^{-sn\pi/3}$$

$$X(s) = \sum_{n=0}^{\infty} (-1)^n \frac{e^{-sn\pi/3}}{s^2 + 4s + 13}$$

invert term by term.

(R13)

$$\mathcal{L}^{-1}\left(\frac{e^{-sn\pi/3}}{s^2+4s+13}\right) = \mathcal{L}^{-1}\left(\frac{e^{-sn\pi/3}}{s^2+4s+4+9}\right)$$

$$\Rightarrow F(s) = \frac{3}{(s+2)^2+3^2}\cdot\frac{1}{3} \Rightarrow f(t) = \frac{1}{3}e^{-2t}\sin 3t.$$

$$\mathcal{L}^{-1}\left(e^{-sn\pi/3}F(s)\right) = u\left(t-\frac{n\pi}{3}\right)f\left(t-\frac{n\pi}{3}\right)$$

$$= \frac{1}{3}u\left(t-\frac{n\pi}{3}\right)e^{-2\left(t-\frac{n\pi}{3}\right)}\underbrace{\sin\left(3(t-n\pi/3)\right)}_{(-1)^n\sin 3t.}$$

$$\cancel{x(t) = \frac{1}{3}e^{-2t}(-1)^n}$$

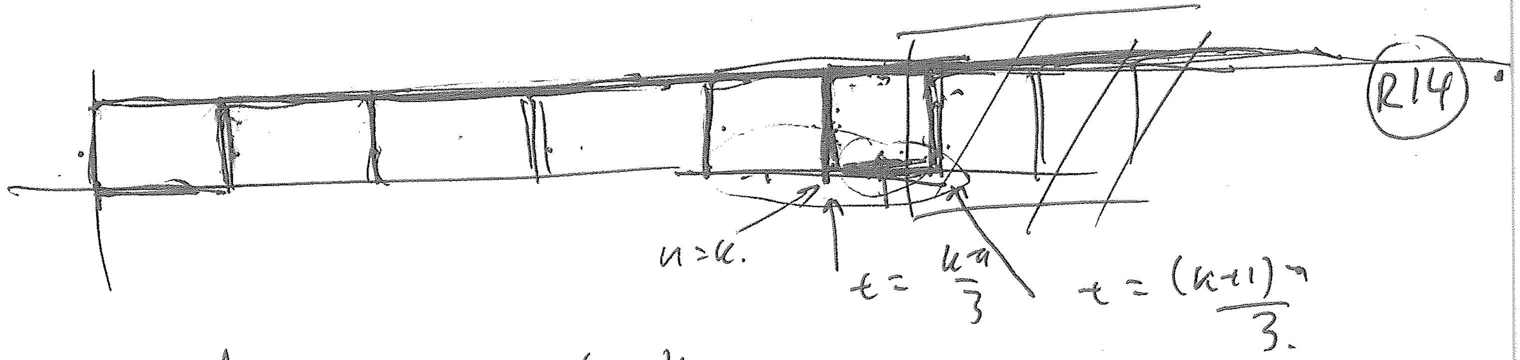
$$= \frac{1}{3}e^{-2t}\sin 3t\,(-1)^n u(t-n\pi/3)\,e^{2n\pi/3}$$

$$x(t) = \frac{1}{3}e^{-2t}\sin 3t\sum_{n=0}^{\infty}(-1)^n(-1)^n u(t-n\pi/3)\,e^{2n\pi/3}$$

$$= \frac{1}{3}e^{-2t}\sin 3t\sum_{n=0}^{\infty}u(t-n\pi/3)\,e^{2n\pi/3}.$$

now on interval

$$\frac{k\pi}{3} < t < \frac{(k+1)\pi}{3}. \qquad t < \frac{(k+1)\pi}{3}.$$



sum only goes up to k. $\quad 1 + 1 + 1 + \dots + 1$

on this interval

$$x(t) = \frac{1}{3} e^{-2t} \sin 3t \boxed{\sum_{n=0}^{k} \cancel{e^{2n\pi/3}} \left(e^{2\pi/3}\right)^n}$$

$$S' = \sum_{n=0}^{k} z^n = 1 + z + \dots + z^n$$

$$z S' = z + \dots z^n + z^{k+1}$$

$$(1-z) S' = 1 - z^{k+1} \Rightarrow S' = \frac{1 - z^{k+1}}{1-z}.$$

here $z = e^{2\pi/3}$.

$$x(t) = \frac{1}{3} e^{-2t} \sin 3t \left[\frac{1 - e^{\frac{2\pi}{3}(k+1)}}{1 - e^{2\pi/3}} \right]$$

note: for $|z| < 1$, $k \to \infty$.

$z^{k+1} \to 0$ as $k \to \infty$.

then $S' = \frac{1}{1-z}$

jump condition for δ-fcn (solve w/o $\mathcal{L}(\cdot)$).

$$mx'' + kx = F\delta(t-a); \quad x(0) = \alpha, \quad x'(0) = \beta.$$

① $\underline{0 < t < a}$: $x = x_l$

$$mx_l'' + kx_l = 0 \qquad x_l(0) = \alpha, \quad x_l'(0) = \beta.$$

$$x_l = A\cos\omega t + B\sin\omega t, \quad \omega^2 = k/m$$

② $t > a$, $x = x_r$

$$mx_r'' + kx_r = 0 \qquad x_r(a) = x_l(a)$$

continuity condition

$$mx'' + kx = F\delta(t-a)$$

for $\varepsilon \ll 1$ $\quad \varepsilon > 0$

$$\int_{a-\varepsilon}^{a+\varepsilon} mx''\,dt + k\boxed{\int_{a-\varepsilon}^{a+\varepsilon} x\,dt} = F\int_{a-\varepsilon}^{a+\varepsilon} \delta(t-a)\,dt.$$

$\approx 2\varepsilon\, x(a) \to 0$ as $\varepsilon \to 0$. (x is finite)

$$m\left[x'\right]_{a-\varepsilon}^{a+\varepsilon} = F \cdot 1$$

$$m\left[x'(a+\varepsilon) - x'(a-\varepsilon)\right] = F$$

as $\varepsilon \to 0$, $\quad x'(a+\varepsilon) = x_r'(a).$

$$x'(a-\varepsilon) = x_l'(a)$$

$$x_r'(a) - x_l'(a) = \frac{F}{m}$$

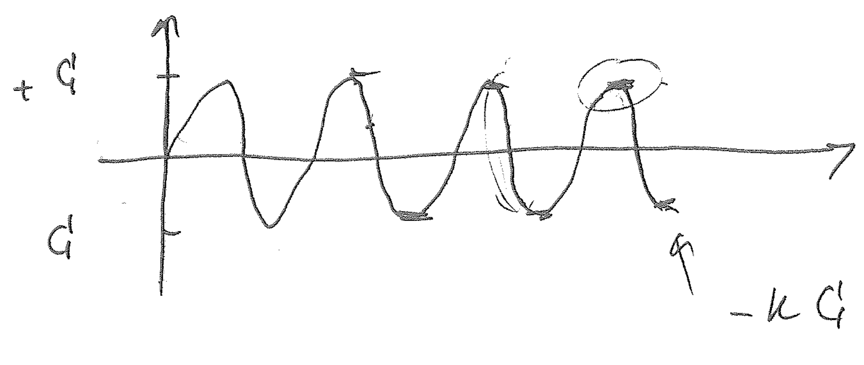
$$x_r'(a) = x_l'(a) + \frac{F}{m}.$$

consider:

$$m x'' + k x = F\,u(t-a) \qquad x(0) = \alpha,\ x'(0) = \beta.$$

choose F, a, to stop all oscillations for $t > a$.

$$kG = F$$



(R17)

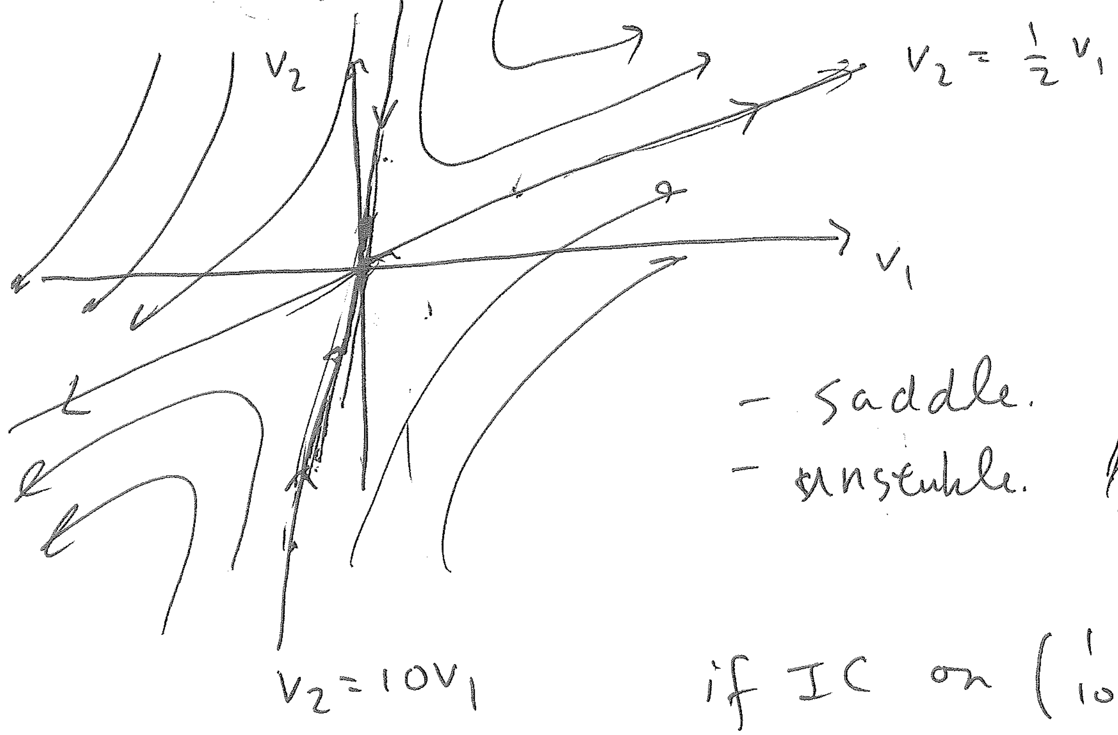
$mx'' + kx = k_0 C_1$, $x(0) = C_1$

$x'(0) = 0$.

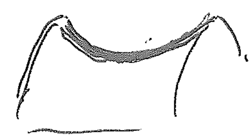
$x = C_1$ solves ODE.

<u>EX</u> plot phase diagram for

$$\vec{v} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 10 \end{pmatrix} e^{-2t}.$$



- saddle.
- unstable.

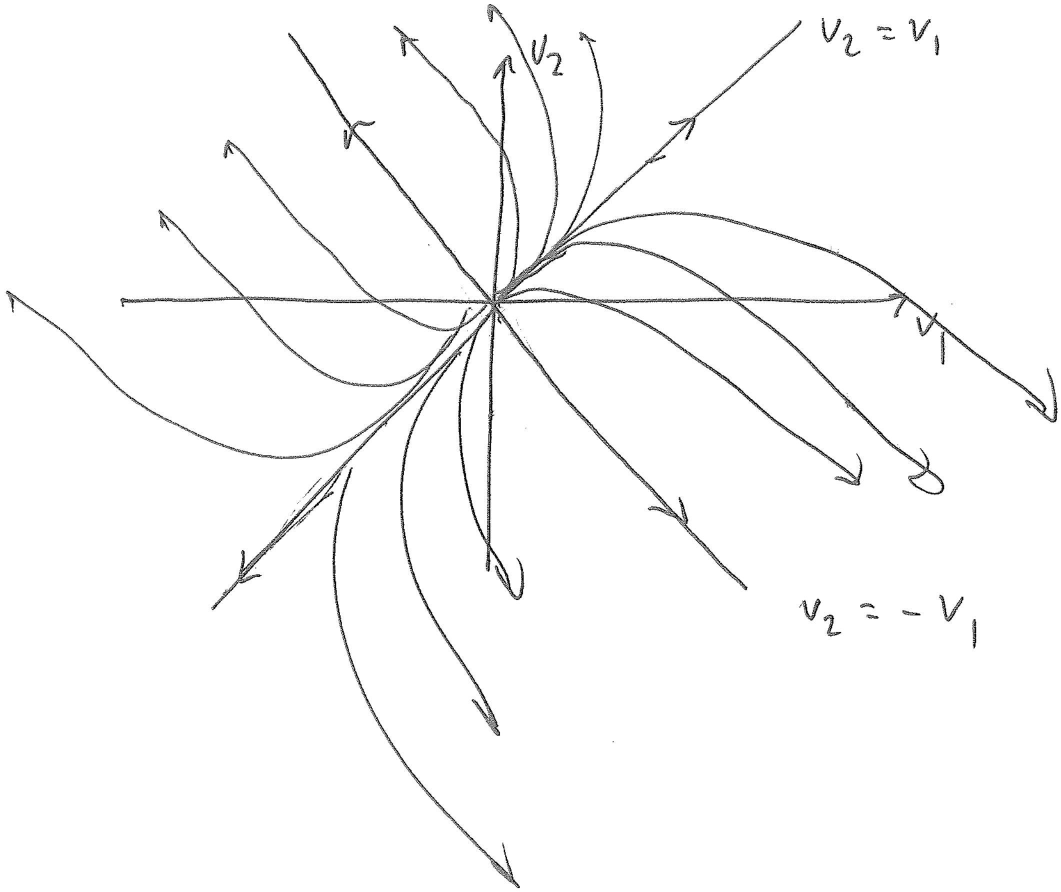


if IC on $\begin{pmatrix} 1 \\ 10 \end{pmatrix}$ then

$c_1 = 0 \to$ origin

<u>EX</u>

plot $\vec{v} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}.$



node
unstable.

if eigenvalues both real, negative, reverse the arrows.