MATH 2120 – Practice Final

- Duration: 3 hours.
- A table of Laplace transforms is included as the final page.
- Use of notes or calculators is not allowed.
- Show all your work and write legibly.
- No credit will be given for memorized answers.
- Only work done in the exam booklet will be graded - if you run out of room, please ask for another and write your name on both booklets. **Work done outside of the booklet will not be graded - you must show all work in the booklet to receive full credit.**
- If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation $y''+p(x)y'+q(x)y = 0$, then a particular solution $y_p(x)$ of the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x)\,,$$

is

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)\,,$$

where

$$c_1(x) = -\int \frac{y_2(x)g(x)}{W(x;y_1,y_2)}\,dx\,, \qquad c_2(x) = \int \frac{y_1(x)g(x)}{W(x;y_1,y_2)}\,dx\,; \qquad W(x;y_1,y_2) \equiv y_1y_2' - y_2y_1'\,.$$

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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

1. Consider the initial value problem

$$\frac{dy}{dt} + \frac{1}{t}y = \log t\,, \qquad t > 0\,, \quad y(1) = y_0\,. \tag{1}$$

   (a) Solve (1).

   (b) For what value of y_0 does the solution behave qualitatively differently as $t \to 0^+$?

2. Find the general solution of

$$\frac{1}{2}x^2(x-1)y'' - x(x-2)y' + (x-3)y = 0\,,$$

   given that $y = x^2$ is a solution.

3. Consider the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 - 6x + 4}{2(y-1)}\,, \qquad y(2) = 1 - \sqrt{2}\,. \tag{2}$$

   (a) Find the unique solution of (2).

   (b) On what interval in x does the solution exist? [To find zeros of a cubic, try, in this order, 0, 1, -1, 2, -2. If none of those work, check your work. If all your work is correct, ask the examiner if there is a typo in the question.]

   (c) What happens if the initial condition was instead $y(2) = 1$? Explain.

4. Solve the initial value problem

$$y'' + y = \begin{cases} t\,, & 0 \le t \le \pi\,, \\ \pi e^{\pi - t}\,, & t > \pi\,, \end{cases} \qquad y(0) = 0, \quad y'(0) = 1\,,$$

   using Laplace transforms. Show that the solution is the same as that obtained by solving it piece-wise in Homework 4.

5. Consider the equation

$$y'' + y = f(t)\,, \qquad t > 0\,, \quad y(0) = y'(0) = 0\,.$$

   Show that the solution by means of the convolution theorem for Laplace transforms is identical to the one obtained using variation of parameters.

6. Use Laplace transforms to solve

$$y'' + \omega_0^2 y = F_0 \cos(2t)\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,,$$

for all possible values of ω_0. The following may be useful:

$$\mathcal{L}\{t\sin(kt)\} = \frac{2sk}{(s^2+k^2)^2}\,, \qquad \mathcal{L}\{t\cos(kt)\} = \frac{s^2-k^2}{(s^2+k^2)^2}\,.$$

7. Consider the initial value problem

$$y'' + 4y = \begin{cases} 0\,, & 0 \le t < 5\,, \\ \frac{t-5}{5}\,, & 5 \le t < 10\,, \\ 1\,, & t > 10\,, \end{cases} \qquad y(0) = 0, \quad y'(0) = 0\,. \tag{3}$$

   (a) Sketch the forcing function.

   (b) Solve (3).

   (c) Explain the behaviour of the solution on $(0,5)$, $(5,10)$ and $(10,\infty)$ in terms of the behaviour of the forcing function. About what mean value does the solution oscillate when $t > 10$?

8. Solve

$$x'' + x = \delta(t - 2\pi)\,, \qquad x(0) = \alpha\,, \qquad x'(0) = \beta\,,$$

both with *and* without using Laplace transforms. Check to see that your answers agree.

9. Explain why $y = (\cos t - 1)/t^3$ does not have a Laplace transform, but $y = (\cos t - 1)/t^2$ does. Compute the Laplace transform of $y = (\cos t - 1)/t$.

10. Consider the system of equations

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

   (a) Find the general solution of (4) and draw a few trajectories in the phase ($x - y$) plane.

   (b) If $x(0) = 5$ and $y(0) = -2$, what value does the ratio y/x approach as $t \to \infty$? Note, you do not have to solve for any constants - a simple explanation may suffice. You may also solve for the constants if you wish.

   (c) On what line in the $x - y$ plane should one set the initial conditions so that $x \to 0$ and $y \to 0$ as $t \to \infty$?

11. Write the second order equation

$$y'' + 9y = 0\,, \qquad y(0) = \alpha\,, \quad y'(0) = \beta\,,$$

as a system of two first order equations. Solve the resulting system.