Existence of solutions of

$$\frac{d\tilde{y}}{d\hat{t}} = f(\hat{t}, \tilde{y}), \quad \tilde{y}(\hat{t}_0) = \tilde{y}_0$$

let $\quad t = \hat{t} - \tilde{t}_0 \quad , y = \tilde{y}_{\bullet} - \tilde{y}_0$

then

$$\frac{dy}{dt} = f(t, y), \quad y(0) = 0 \qquad (1)$$

<u>assumptions</u> : $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous in some rectangle D around point $(0,0)$

let $y = \phi(t)$ be a solution to (1)

then

$$\frac{d\phi}{dt} = \underbrace{f(t, \phi(t))}_{\text{function only of } t.}$$

so we can write

$$\phi(t) = \int_0^t f(t, \phi(t))\, dt \qquad (2)$$

[note: $\phi(0) = 0$ is satisfied]

we can solve this integral equation (2) by an iterative procedure:

$$\phi_0(t) = 0$$

$$\phi_1(t) = \int_0^t f(t, \phi_0(t))\, dt$$

$$\vdots$$

$$\phi_{n+1}(t) = \int_0^t f(t, \phi_n(t))\, dt$$

if ~~$\lim_{n\to\infty} \phi_n$~~ $\{\phi_n\}$ converges as $n \to \infty$,

then $\phi(t)$ is solution (~~$\lim_{n\to\infty}$~~ $\phi(t) = \lim_{n\to\infty} \phi_n(t)$)

since then

$$\phi(t) = \lim_{n\to\infty} \int_0^t f(t, \phi_n(t))\, dt$$

(E42

$$= \int_0^t \lim_{n\to\infty} f(t, \phi_n(t))\, dt$$

allowed if $\{\phi_n\}$ converges uniformly

$$= \int_0^t \cancel{\lim_{n\to\infty}} f(t, \lim_{n\to\infty} \phi_n(t))\, dt$$

allowed if $f(t,y)$ continuous in second variable

$$= \int_0^t f(t, \phi(t))\, dt$$

then by fundamental theorem of calculus,

$$\frac{d\phi}{dt} = f(t, \phi(t)), \quad \phi(0) = 0.$$

now just need to establish convergence of $\{\phi_n(t)\}$

write $\phi_n(t)$ as a <u>telescopic series</u>

$$\phi_n = \phi_1 + [\phi_2 - \phi_1] + \dots + [\phi_n - \phi_{n-1}]$$

which is the n^{th} partial sum of the series

$$\phi_1(t) + \sum_{j=1}^{\infty} [\phi_{j+1}(t) - \phi_j(t)] \quad (*)$$

so $\{\phi_n\}$ converges if $(*)$ converges.

$(*)$ converges if

$$|\phi_1(t)| + \sum_{j=1}^{\infty} |\phi_{j+1}(t) - \phi_j(t)| \text{ converges}$$

(direct comparison test)

so we need to estimate $|\phi_{j+1} - \phi_j|$.

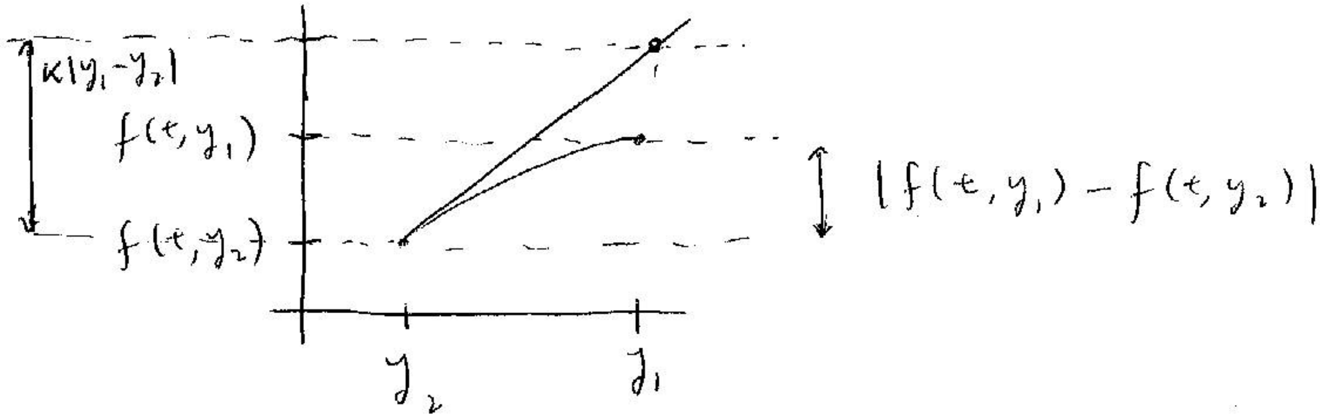
first, we note that if $\frac{\partial f}{\partial y}$ is continuous in the rectangle D,

$$\left|\frac{\partial f}{\partial y}\right| \leq K$$

in D for some finite $K > 0$.

this implies

$$|f(t,y_1) - f(t,y_2)| \leq K\,|y_1 - y_2|$$



so we have

$$\left| f(t, \phi_n(t)) - f(t, \phi_{n-1}(t)) \right| \leq K\,\left| \phi_n(t) - \phi_{n-1}(t) \right|$$

+

now

$$|\phi_1(t)| = \left| \int_0^t f(t, \phi_0(t))\, dt \right|$$

since f is ~~bounded~~ continuous on D,

$|f(t,y)| \leq M$ in D for some finite $M \geq 0$.

$$\Rightarrow |\phi_1(t)| \leq \int_0^t |f(t, \phi_0(t))|\, dt$$

$$\leq \int_0^t M\, dt$$

$$\leq M|t|$$

now

$$\phi_2 - \phi_1 = \int_0^t f(t, \phi_1)\, dt - \int_0^t f(t, \phi_0)\, dt$$

$$|\phi_2 - \phi_1| \leq \int_0^t |f(t, \phi_1) - f(t, \phi_0)|\, dt$$

$$\leq k \int_0^t |\phi_1 - \phi_0|\, dt \quad (\text{by } \dagger)$$

$$\phi_0$$

$$= k \int_0^t |\phi_1|\, dt$$

$$\leq k \int_0^t M t\, dt = kM\frac{t^2}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| \leq kM\frac{t^2}{2}$$

induction to obtain bound on $|\phi_n - \phi_{n-1}|$:

assume

$$|\phi_{n-1} - \phi_{n-2}| \le K^{n-2} M \frac{|t|^{n-1}}{(n-1)!}$$

$$\phi_n - \phi_{n-1} = \int_0^t f(t, \phi_{n-1}) - f(t, \phi_{n-2})\, dt$$

$$|\phi_n - \phi_{n-1}| \le \int_0^t |f(t, \phi_{n-1}) - f(t, \phi_{n-2})|\, dt$$

$$\le K \int_0^t |\phi_{n-1} - \phi_{n-2}|\, dt$$

$$\le K \int_0^t K^{n-2} M \frac{t^{(n-1)}}{(n-1)!}\, dt$$

$$= K^{n-1} M \frac{|t|^n}{n!}$$

$$\Rightarrow \quad |\phi_n - \phi_{n-1}| \le K^{n-1} M \frac{|t|^n}{n!}$$

$$|\phi_{j+1} - \phi_j| \le K^j M \frac{|t|^{j+1}}{(j+1)!}$$

ratio test for series convergence:

$$\lim_{j\to\infty} \frac{|\phi_{j+1} - \phi_j|}{|\phi_j - \phi_{j-1}|} = \lim_{j\to\infty} \frac{K^j M |t|^{j+1}}{(j+1)!} \cdot \frac{j!}{K^{j-1} M |t|^j}$$

$$= \lim_{j\to\infty} \frac{K|t|}{j} < 1$$ for t inside rectangle D.

so (*) converges $\Rightarrow$ $\{\phi_n\}$ converges.

Uniqueness

let $\phi(t)$ and $\psi(t)$ both be solutions to (1). Then

$$\phi(t) - \psi(t) = \int_0^t f(t,\phi) - f(t,\psi)\,dt.$$

$$|\phi(t) - \psi(t)| \le \int_0^t |f(t,\phi) - f(t,\psi)|\,dt$$

$$\le K\int_0^t |\phi - \psi|\,dt.$$

let $u(t) = \int_0^t |\phi(t) - \psi(t)|\, dt$.

so that $u(0) = 0$, $u(t) \geq 0$ for $t \geq 0$.

also, $\frac{du}{dt} = |\phi(t) - \psi(t)|$

$$\Rightarrow \frac{du}{dt} \leq Ku \Rightarrow \frac{du}{dt} - Ku \leq 0.$$

or

$$\frac{d}{dt}\left(e^{Kt} u\right) \leq 0$$

$$\Rightarrow e^{Kt} u(t) \leq 0 \quad \text{for } t \geq 0.$$

since $e^{Kt} \geq 0$ for all $t \geq 0$,

$$u(t) \leq 0 \quad \text{for } t \geq 0.$$

since we said $u(t) \geq 0$, this means $u(t) \equiv 0$ for all $t \geq 0$, and $u'(t) \equiv 0$

$$\Rightarrow \phi(t) = \psi(t)$$

so only one solution