MATH 2120 – Homework 1 (due Thursday September 25, 2014 before class)

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1. §1.4 (p. 43) #4, #17.

2. Solve the initial value problem
$$y'(x) = \frac{xy^3}{\sqrt{1+x^2}}\,, \qquad y(0) = 1\,,$$
and determine the interval on which the solution exists.

3. Solve the initial value problem
$$\frac{dy}{dx} = 2y^2 + xy^2\,, \qquad y(0) = 1\,. \tag{1}$$
Determine the interval on which the solution exists. On this interval, the solution has an extremum value – determine its location and whether it is a minimum or maximum (hint: recall the second derivative test from first year – there is an easy way to determine this from (1)).

4. §1.5 (p. 54) #4, #7, #16.

5. Solve the initial value problem
$$\frac{dy}{dt} - \frac{1}{2}y = 2\cos t\,, \qquad y(0) = a\,.$$
For what value(s) of a does the solution (a) remain finite for all time, (b) approach $-\infty$ as $t \to \infty$ and (c) approach $+\infty$ as $t \to \infty$?

6. **Optional bonus question:** The first person to provide a fully justified answer (as determined by me) to all parts of this problem will have the option of (a) dropping their second worst quiz grade in addition to their worst, or (b) getting the full 5% homework credit without handing in any homeworks (choose this option at your own peril). Please ask for clarifications on any part of the problem if required. This will remain open for at least a month unless someone gets the answer before then.

   This question shows that we can still obtain useful properties of a solution even though we are unable to find an explicit expression for it.

   (a) Find an implicit solution to the differential equation
   $$\frac{dy}{dx} = \frac{1}{x+y}\,, \qquad y(x_0) = y_0\,.$$
   [hint: solve for $x = x(y)$]

(b) Find the inequality relation (i.e., find $f(x_0)$)

$$y_0 > f(x_0) \tag{2}$$

between x_0 and y_0 for which the solution $y(x)$ has a vertical tangent at some point (x^*, y^*). Calculate x^* and y^* in terms of x_0 and y_0.

(c) The presence of a vertical tangent (or "saddle node") indicates that the solution $y(x)$ only exists when $x < x^*$ or $x > x^*$. Show that it is the latter.

(d) When the inequality (2) is satisfied (i.e., there is a saddle node), find the range of y_0 for which $y \to +\infty$ as $x \to \infty$, and the range of y_0 for which $y \to -\infty$ as $x \to \infty$. In both cases, determine the behavior of $y(x)$ as $x \to \infty$.

(e) When the inequality (2) is reversed (i.e., there is no saddle node), the solution exists for all x. Determine the behaviors of $y(x)$ as $x \to -\infty$ and $+\infty$.