MATH 2120 – Homework 2 (due Tuesday October 7, 2014 before class)

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1. **[OPTIONAL]** Stability of equilibrium solutions: for first order autonomous equations, which have the form

$$\frac{dy}{dt} = f(y)\,, \tag{1}$$

stability of equilibrium solutions is of interest. A solution $y = y_0$ is an equilibrium solution if $f(y_0) = 0$, since then $dy/dt = 0$. Here, y_0 is a constant. An equilibrium is **stable** if small perturbations away from it decay as time increases so that the solution returns to the equilibrium. It is **unstable** if small perturbations grow in time so that the solution evolves away from the equilibrium. That is, if we let

$$y = y_0 + \delta(t)\,, \tag{2}$$

with $|\delta(0)| \ll 1$ being the initial perturbation, y_0 is a stable equilibrium if $\delta(t)$ decreases in time, and unstable if $\delta(t)$ increases in time. Substitute (2) into (1) and use Taylor series to obtain a **linear** ODE for $\delta(t)$. Show that y_0 is a stable equilibrium if $f'(y_0) < 0$ and unstable if $f'(y_0) > 0$. Use this result to determine the stability of the two equilibria of the logistic growth equation where $f(y) = y(1 - y)$.

2. Section 1.6 (p. 72): #1, #2, #3, #10, #11. For these questions, **DO NOT SOLVE** the ODE. Simply make the substitution $v(x) = y(x)/x$ and obtain a separable equation for $v(x)$.

3. Find the general solution of

$$\frac{dy}{dx} - \frac{3}{4x}y = x^4 y^{1/3}\,.$$

4. (a) Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to show the identity

$$\sin(a+b) = \sin a\cos b + \sin b\cos a$$

(a similar identity can be derived for $\cos(a+b)$ using the same approach).

(b) Use Euler's formula to show the identity

$$\sin^3 t = \frac{1}{4}\left(3\sin t - \sin 3t\right)$$

(a similar identity can be derived for $\cos^3 t$).

(c) Use Euler's formula to show that

$$\frac{d}{dx}\cos mx = -m\sin mx\,,$$

and

$$\frac{d}{dx}\sin mx = m\cos mx\,.$$

(d) **Definition of** $\arcsin x$**:** If

$$\sin\theta = x\,, \tag{3}$$

then $\theta = \arcsin x$. So if we can solve for θ, we would know what the arcsine function actually is. To do so, use Euler's formula to write $\sin\theta$ in (3) in terms of complex exponentials. Then define $w = e^{i\theta}$ so that $e^{-i\theta} = 1/w$. Solve the resulting equation for w (it will be a quadratic equation). Then solve for θ. The positive and negative branches of the square root correspond to two different possible values of θ. For example, if $\sin\theta = 1/\sqrt{2}$, then two possible values of θ are $\theta = \pi/4$ and $\theta = 3\pi/4$. Notice also that any integer multiple of 2π can be added to θ; this reflects the multi-valued nature of the logarithmic function.

(e) Use the definitions $\cosh x \equiv (e^x + e^{-x})/2$ and $\sinh x \equiv (e^x - e^{-x})/2$ to show that

$$\frac{d}{dx}\cosh mx = m\sinh mx\,,$$

and

$$\frac{d}{dx}\sinh mx = m\cosh mx\,.$$

(f) Show that $\cosh^2 mx - \sinh^2 mx = 1$ for any x.

5. Consider the IVP

$$y'' - 9y = 0\,; \qquad y(0) = \alpha\,, \quad y'(0) = \beta\,. \tag{4}$$

(a) Write the general solution as $y = c_1 e^{3t} + c_2 e^{-3t}$ and solve for c_1 and c_2 using the initial conditions.

(b) Write the general solution as $y = a_1\cosh 3t\, +\, a_2\sinh 3t$ and solve for a_1 and a_2 using the initial conditions. Show that your solutions in parts (a) and (b) are equivalent. Which way made the algebra simpler?

(c) If the initial conditions were instead $y(t_0) = \alpha$ and $y'(t_0) = \beta$, what would you do to simplify the algebra?

6. Solve the IVP

$$y'' + 5y = 0\,; \qquad y(2) = \alpha\,, \quad y'(2) = \beta\,.$$

7. Solve the IVP

$$y'' + 2y' + 5y = 0\,; \qquad y(0) = 1\,, \quad y'(0) = 0\,.$$

8. Consider the "equidimensional" equation

$$x^2y'' + axy' + by = 0\,. \tag{5}$$

For $a = -3$, and $b = 3$, show that $y_1 = x$ is a solution of (5). Use reduction of order to find the other linearly independent solution y_2. For $x \neq 0$, show that the Wronskian $W(y_1, y_2)$ is nonzero.