# Homework #2 solutions

1. $$\frac{dy}{dt} = f(y)$$

let $f(y_0) = 0$ and perturb $y$ from $y_0$:

$$y = y_0 + \delta(t) \qquad \text{for} \qquad |\delta(t)| \ll 1$$

then

$$\frac{d\delta}{dt} = f(y_0 + \delta(t))$$
$$= \cancel{f(y_0)} + f'(y_0)\,\delta(t)$$

$$\frac{d\delta}{dt} = f'(y_0)\,\delta$$

so

$$\delta(t) = \delta(0)\, e^{f'(y_0)t}$$

so if $f'(y_0) > 0$, then $y_0$ is unstable since $\delta(t)$ increases in time if $f'(y_0) < 0$, $\delta(t)$ decays into so

$y_0$ is ~~stab~~ stable

for $f(y) = y(1-y) = y - y^2$, the

equilibria are $y_1 = 0$, $y_2 = 1$.

$$f'(y) = 1 - 2y$$

$$\boxed{\begin{array}{l} f'(0) = 1 \Rightarrow 0 \text{ is unstable.} \\ f'(1) = 1-2 = -1 \Rightarrow 1 \text{ is stable.} \end{array}}$$

(2) Section 1.6 #1

$$y' = \frac{x-y}{x+y} = \frac{1 - y/x}{1 + y/x}.$$

let $v = y/x \Rightarrow y = xv \Rightarrow y' = v + xv'$

then

$$v + xv' = \frac{1-v}{1+v}$$

$$xv' = \frac{1-v}{1+v} - v$$

$$\boxed{v' = \frac{1}{x}\left[\frac{1-v}{1+v} - v\right]}$$

Section 1.6 #2

$$y' = \frac{x^2+2y^2}{2xy} = \frac{1}{2}\left[\frac{x}{y} + \frac{y}{x}\right]$$

let $y = xv$

then

$$v + xv' = \frac{1}{2}\left[\frac{1}{v} + v\right].$$

$$xv' = \frac{1}{2}\left[\frac{1}{v} + v\right] - v.$$

$$\boxed{\frac{dv}{dx} = \frac{1}{x}\left[\frac{1}{2}\left(\frac{1}{v} + v\right) - v\right]}$$

section 1.6 #3

$$\frac{dy}{dx} = \frac{1}{x}\left(y + 2\sqrt{xy}\right)$$

$$\frac{dy}{dx} = \frac{y}{x} + 2\sqrt{y/x} \qquad \text{let } y = xv$$

then

$$v + xv' = v + 2\sqrt{v}$$

$$xv' = 2\sqrt{v}$$

$$\boxed{v' = \frac{2}{x}\sqrt{v}}$$

section 1.6 #10

$$y' = \frac{x^2 + 3y^2}{xy} = \frac{x}{y} + \frac{3y}{x}.$$

$y = xv$

$$\Rightarrow \quad v + xv' = \frac{1}{v} + 3v.$$

$$xv' = \frac{1}{v} + 2v.$$

$$\boxed{v' = \frac{1}{x}\left[\frac{1}{v} + 2v\right]}$$

Section 1.6 #11

$$y' = \frac{2xy}{x^2-y^2} = \frac{2}{\frac{x}{y} - \frac{y}{x}}$$

let $y = xv$

then

$$v + xv' = \frac{2}{\frac{1}{v} - v}$$

$$xv' = \frac{2}{\frac{1}{v} - v} - v$$

$$\boxed{v' = \frac{1}{x}\left[\frac{2}{\frac{1}{v} - v} - v\right]}$$

(3) $\frac{dy}{dx} - \frac{3}{4x} y = x^4 y^{1/3}$

(Bernoulli eqn with $n = 1/3$)

let $v = y^{1-n} = y^{2/3}$ or $y = v^{3/2}$.

$$\frac{dy}{dx} = \frac{3}{2} v^{1/2} \frac{dv}{dx}$$

so

$$\frac{3}{2} v^{1/2} \frac{dv}{dx} - \frac{3}{4x} v^{3/2} = x^4 v^{1/2}$$

$$\frac{dv}{dx} - \frac{1}{2x} v = \frac{2}{3} x^4 \qquad \text{(linear eqn)}$$

integrating factor is $e^{-\frac{1}{2}\log x} = \frac{1}{\sqrt{x}}$

so

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}} v\right) = \frac{2}{3} x^{7/2}$$

$$\frac{1}{\sqrt{x}} v = \frac{2}{3} \cdot \frac{2}{9} x^{9/2} + c$$

$$v = \frac{4}{27} x^5 + c\sqrt{x}$$

finally, with $v = y^{2/3}$, or $y = v^{3/2}$.

$$\boxed{y = \left(\frac{4}{27} x^5 + c\sqrt{x}\right)^{3/2}}$$

4 a) show $\sin(a+b) = \sin a \cos b + \sin b \cos a$

$$e^{i(a+b)} = e^{ia} e^{ib} = (\cos a + i \sin a)(\cos b + i \sin b)$$

$$= \cos a \cos b - \sin a \sin b \qquad (1)$$

$$+ i [\sin a \cos b + \sin b \cos a].$$

but $e^{i(a+b)} = \cos(a+b) + i \sin(a+b) \qquad (2)$

equate real and imag. parts of

(1) and (2):

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$

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b) show $\sin^3 t = \frac{1}{4}(3 \sin t - \sin 3t)$

$$\sin^3 t = \left[\frac{1}{2i}\left(e^{it} - e^{-it}\right)\right]^3$$

$$= \frac{-1}{8i}\left[e^{i3t} - 3e^{it} + 3e^{-it} - e^{-i3t}\right]$$

$$= \frac{-1}{8i}\left[ e^{i3t} - e^{-i3t} - 3\left(e^{it} - e^{-it}\right) \right]$$

$$= \frac{-1}{8i}\left[ 2i\sin 3t - 3\cdot 2i\sin t \right]$$

$$= \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$$

$$= \frac{1}{4}\left[ 3\sin t - \sin 3t \right]$$

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c) show $\frac{d}{dx}\cos mx = -m\sin mx$

$$\frac{d}{dx}\sin mx = m\cos mx.$$

$$e^{imx} = \cos mx + i\sin mx$$

$$\frac{d}{dx}e^{imx} = \frac{d}{dx}\cos mx + i\frac{d}{dx}\sin mx$$

$$im\,e^{imx} = \frac{d}{dx}\cos mx + i\frac{d}{dx}\sin mx.$$

$$im(\cos mx + i\sin mx) = \frac{d}{dx}\cos mx + i\frac{d}{dx}\sin mx$$

equate real and imag. parts

$$\Rightarrow \frac{d}{dx} \cos mx = - m \sin mx$$

$$\frac{d}{dx} \sin mx = m \cos x$$

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d) $\sin \theta = x$

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = x$$

let $w = e^{i\theta}$. so

$$w - \frac{1}{w} = 2ix$$

$$w^2 - 1 = 2ixw$$

$$w^2 - 2ixw - 1 = 0$$

$$w = \frac{2ix \pm \sqrt{-4x^2 + 4}}{2}$$

$$= ix \pm \sqrt{1 - x^2}$$

then $e^{i\theta} = ix \pm \sqrt{1-x^2}$

$\Rightarrow$ $\alpha =$ ~~$\frac{1}{i}$~~

$$i\theta = \log\left[ix \pm \sqrt{1-x^2}\right].$$

$$\boxed{\theta = -i\log\left[ix \pm \sqrt{1-x^2}\right].}$$

e) $\cosh mx = \dfrac{e^{mx}+e^{-mx}}{2}$, $\sinh mx = \dfrac{e^{mx}-e^{-mx}}{2}$

$$\frac{d}{dx}\cosh mx = \frac{me^{x}-me^{-x}}{2} = m\sinh mx$$

$$\frac{d}{dx}\sinh mx = \frac{me^{mx}+me^{-mx}}{2} = m\cosh mx.$$

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(5) a) $y''-9y=0$, $y(0)=\alpha$, $y'(0)=\beta$.

$$y = c_1e^{3t}+c_2e^{-3t}.$$

$y(0) = c_1 + c_2 = \alpha.$ $\Rightarrow$ $c_1+c_2=\alpha$

$y'(0) = 3c_1-3c_2=\beta$ $\Rightarrow$ $c_1-c_2=\beta/3.$

add the 2 equations:

$$2c_1 = \alpha + \beta/3$$

$$\Rightarrow c_1 = \frac{1}{2}\left[\alpha + \beta/3\right].$$

subtract:

$$2c_2 = \alpha - \beta/3$$

$$c_2 = \frac{\alpha - \beta/3}{2}.$$

then

$$\boxed{y = \frac{1}{2}(\alpha + \beta/3)e^{3t} + \frac{1}{2}(\alpha - \beta/3)e^{-3t}}$$

b) $y = a_1 \cosh 3t + a_2 \sinh 3t.$

$y(0) = a_1 = \alpha \Rightarrow a_1 = \alpha.$

$y'(0) = 3a_2 = \beta \Rightarrow a_2 = \beta/3$

$$\Rightarrow y = \alpha \cosh 3t + \frac{\beta}{3}\sinh 3t.$$

$$= \alpha\left[\frac{e^{3t} + e^{-3t}}{2}\right] + \frac{\beta}{3}\left[\frac{e^{3t} - e^{-3t}}{2}\right]$$

$$= \frac{1}{2}\left[\alpha + \beta/3\right] e^{3\tau} + \frac{1}{2}\left[\alpha - \beta/3\right] e^{-3\tau}$$

so equivalent to part a).

c) if IC's posed as $y(t_0) = \alpha$,

$y'(t_0) = \beta$, we make a transform:

$y(t) = z(t - t_0) = z(\tau)$ where

$\tau = t - t_0$. | note: when $t = t_0$, $\tau = 0$

then

$$\frac{dy}{dt} = \frac{dz}{d\tau}\frac{d\tau}{dt} = \frac{dz}{d\tau}$$

so

$$\frac{d^2y}{dt^2} = \frac{d^2z}{d\tau^2}$$

$$\Rightarrow \frac{d^2z}{d\tau^2} = 9z, \quad z(0) = \alpha, \quad z'(0) = \beta$$

then $z = \alpha\cosh 3\tau + \frac{\beta}{3}\sinh 3\tau$

$$y = z(t - t_0)$$

$$= \alpha \cosh(3(t - t_0)) + \frac{\beta}{3} \sinh(3(t - t_0))$$

another way to think about it:

$$y'' = 9y$$

$$\Rightarrow y_1 = e^{3t}, \quad y_2 = e^{-3t}.$$

$$\hat{y}_1 = e^{3t} \cdot \underbrace{e^{-3t_0}}_{\text{just a number}} \qquad \hat{y}_2 = e^{-3t} \underbrace{e^{3t_0}}_{\text{just a number}}$$

so $\hat{y}_1$ and $\hat{y}_2$ are also solutions.

then

$$\frac{\hat{y}_1 + \hat{y}_2}{2} = \cosh(3(t - t_0))$$

and

$$\frac{\hat{y}_1 - \hat{y}_2}{2} = \sinh(3(t - t_0))$$

are also solutions ...

6 $y'' + 5y = 0, \quad y(2) = \alpha, \; y'(2) = \beta$

same trick as 5(c):

$$y = a_1 \cos(\sqrt{5}(x-2)) + a_2 \sin(\sqrt{5}(x-2))$$

$y(2) = \alpha \Rightarrow a_1 = \alpha$

$y'(2) = \beta \Rightarrow \sqrt{5}\, a_2 = \beta \Rightarrow a_2 = \beta/\sqrt{5}.$

then

$$\boxed{y = \alpha \cos(\sqrt{5}(x-2)) + \frac{\beta}{\sqrt{5}} \sin(\sqrt{5}(x-2))}$$

7 $y'' + 2y' + 5y = 0, \quad y(0) = 1, \; y'(0) = 0.$

C.E.: $r^2 + 2r + 5 = 0$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

so $y = e^{-t}\left[A\cos 2t + B\sin 2t\right].$

$y(0) = 1 \Rightarrow \quad A = 1$

$y'(0) = 0 \Rightarrow \quad -(A) + 2B = 0$

$\Rightarrow 2B - 1 = 0 \Rightarrow B = \frac{1}{2}.$

so

$$\boxed{y = e^{-t}\left[\cos 2t + \frac{1}{2}\sin 2t\right]}$$

(8) $x^2 y'' - 3xy' + 3y = 0.$

$y_1 = x, \quad y_2 = x\,v(x)$

$y_2' = v + xv'$

$y_2'' = v' + v' + xv''$

~~$x=1, x''=$~~

$\frac{d}{dx}x = 1, \quad \frac{d^2}{dx^2}x = 0$

then

$0 - 3x \cdot 1 + 3x = 0$

so $y=x$ is a soln

$\Rightarrow x^2[xv'' + 2v'] - 3x[\cancel{v} + xv'] \cancel{+3xv} = 0$

$xv'' + 2v' - 3v' = 0$

$xv'' = v' \qquad$ let $u = v'$

then we have first order eqn for $u$.

$$x u' = u$$

$$\frac{du}{dx} = \frac{u}{x}$$

$$\int \frac{du}{u} = \int \frac{dx}{x}$$

$$\log u = \log x + c$$

$c = 0$ wlog.

so $u = x$

then $v = \int u\, dx = \frac{x^2}{2} + c$

$c = 0$ wlog.

then

$$y_2 = x \cdot \frac{x^2}{2} = \frac{1}{2} x^3$$

can remove prefactor

$$\to \boxed{y_2 = x^3}$$

now

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= x \cdot 3x^2 - x^3 \cdot 1 = 3x^3 - x^3 = 2x^3 \neq 0$$

when $x \neq 0$

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