Homework 3 solutions

(1) $y'' + 2ay' + a^2 y = 0$

a) $r^2 + 2ar + a^2 = 0$

$(r+a)^2 = 0 \Rightarrow r = -a, -a$

so $e^{-at} = y_1$ must be a solution.

b) for $y'' + p(t) y' + q(t) y = 0$

the Wronskian W satisfies

$$\frac{dW}{dt} = -p(t) W$$

here, $p(t) = +2a$

$$\Rightarrow \frac{dW}{dt} = -2aW \Rightarrow W = c_1 e^{-2at}$$

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c) W defined as

$$W = y_1 y_2' - y_2 y_1' = c_1 e^{-2at}$$

from (b)

we have $y_1 = e^{-at}$

solve for ~~y_2~~ y_2:

$$e^{-at} y_2' - y_2\left[-a e^{-at}\right] = c_1 e^{-2at}$$

$$y_2' + a y_2 = c_1 e^{-at}$$

linear equation: use integrating factor:

$$u(t) = e^{at}.$$

$$\frac{d}{dt}\left[e^{at} y_2\right] = c_1 \cancel{e^{t}}$$

$$e^{at} y_2 = c_1 t + c_2$$

$$y_2 = c_1 t e^{-at} + \underbrace{c_2 e^{-at}}$$

this is y_1

so take $c_2 = 0$ wlog.

$$\Rightarrow y_2 = t e^{-at}$$

(2) r_1, r_2 distinct roots of

$$ar^2 + br + c = 0$$

corresponding to

$$ay'' + by' + cy = 0.$$

so $y_1 = e^{r_1 t}$ is one soln.

let
$$y_2 = \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1}$$
(also soln)

let $r_2 = r_1 + \varepsilon$. then

$$y_2 = \frac{e^{(r_1+\varepsilon)t} - e^{r_1 t}}{\varepsilon}$$

$$= \frac{e^{r_1 t}\left[e^{\varepsilon t} - 1\right]}{\varepsilon}$$

let $\varepsilon \ll 1$

$$= \frac{e^{r_1 t}\left[1 + \varepsilon t + \ldots - 1\right]}{\varepsilon}$$

take $\varepsilon \to 0$

then

$$y_2 = e^{r_1 t} \frac{\varepsilon t}{\varepsilon} = t e^{r_1 t}.$$

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(3) <u>section 2.2 # 41</u>

$$(x+1)y'' - (x+2)y' + y = 0 \quad x > -1, \; y_1 = e^x$$

$$y_2 = v(x) y_1 \qquad y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + 2v' y_1' + v y_1''$$

$$(x+1)\left[v'' y_1 + 2v' y_1' \cancel{+ v y_1''} \right] - (x+2)\left[v' y_1 \cancel{+ v y_1'} \right]$$

$$\cancel{+ v y_1} = 0 .$$

$$(x+1)\left[v'' \cancel{e^x} + 2v' \cancel{e^x} \right] - (x+2) v' \cancel{e^x} = 0.$$

$$\cancel{xx} \; (x+1) v'' + (2x+2) v' - (x+2) v' = 0$$

$$(x+1)v' + xv' = 0$$

$$v'' + \frac{x}{x+1}v' = 0$$

let $u = v'$

$$u' = \frac{-x}{x+1}u.$$

$$\int \frac{du}{u} = \int \frac{-x}{x+1}\,dx$$

$$-\int \frac{x}{x+1}\,dx = -\int \frac{x+1-1}{x+1}\,dx$$

$$= -\int 1 - \frac{1}{x+1}\,dx$$

$$= \int -1 + \frac{1}{x+1}\,dx$$

$$= -x + \log(x+1)$$

then $\log u = -x + \log(x+1)$

$$u = e^{-x} e^{\log(x+1)} = (x+1)e^{-x}$$

$$v = \int u\,dx = \int (x+1)e^{-x}\,dx$$

$$= -(x+1)e^{-x} + \int e^{-x}\,dx$$

$$= -(x+1)e^{-x} - e^{-x}$$

$$= -xe^{-x} - 2e^{-x}$$

finally

$$y_2 = \cancel{u}v(x)e^{x} = -x-2 = -(x+2)$$

$$\Rightarrow \boxed{y_2 = x+2}$$

or $-(x+2)$ is fine too.

Section 2.2 #43

$$(1-x^2)y'' - 2xy' + 2y = 0$$

$$y_2 = vx, \quad y_2' = xv' + v, \quad y_2'' = xv'' + 2v'$$

$$(1-x^2)(xv''+2v') - 2x(xv' \cancel{+v}) \cancel{+2xv} = 0$$

$$(1-x^2)xv'' + 2(1-x^2)v' - 2x^2v' = 0$$

$$(1-x^2)xv'' + (2-4x^2)v' = 0$$

let $u = v'$

$$(1-x^2)xu' + (2-4x^2)u = 0$$

$$(1-x^2)x\frac{du}{dx} = (4x^2-2)u$$

$$\int \frac{du}{u} = \int \frac{4x^2-2}{x(1+x)(1-x)}$$

partial fractions:

$$\frac{4x^2-2}{x(1+x)(1-x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}.$$

$$\Rightarrow 4x^2 - 2 = A(1-x^2) + Bx(1+x) + Cx(1-x).$$

$\underline{x=1}$: $\quad 2 = 2B \Rightarrow B = 1.$

$\underline{x = -1}$

$$2 = -2c \Rightarrow c = -1$$

$\underline{x = 0}$ $\quad A = -2$

$$\Rightarrow \int \frac{4x^2 - 2}{x(1-x^2)}\, dx = \int \frac{-2}{x} + \frac{1}{1-x} - \frac{1}{1+x}\, dx$$

$$= -2\log x - \log(1-x) - \log(1+x)$$

$$= \log \frac{1}{x^2(1-x)(1+x)}$$

so we have

$$\log u = \log \frac{1}{x^2(1-x)(1+x)}$$

$$u = \frac{1}{x^2(1-x)(1+x)}$$

now $v = \int u\, dx$:

partial fractions ~~[illegible]~~ :

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{1+x}. = \frac{1}{x^2(1-x)(1+x)}$$

so

$$Ax(1-x^2) + B(1-x^2) + Cx^2(1+x)$$

$$+ Dx^2(1-x) = 1$$

$x=0$:

$$B = 1$$

$x=1$:

$$2C = 1 \Rightarrow C = \frac{1}{2}.$$

$x=-1$: $\quad 2D = 1 \Rightarrow D = \frac{1}{2}.$

coff of x : $\quad A = 0$

$$\Rightarrow \int \frac{dx}{x^2(1-x)(1+x)} = \int \frac{1}{x^2} + \frac{1}{2}\frac{1}{1-x} + \frac{1}{2}\frac{1}{1+x}\,dx$$

$$= -\frac{1}{x} - \frac{1}{2}\log(1-x) + \frac{1}{2}\log(1+x)$$

$$= -\frac{1}{x} + \frac{1}{2}\log\frac{1+x}{1-x}$$

$$y_2 = xV = -1 + \frac{1}{2}x\log\left(\frac{1+x}{1-x}\right)$$

or

$$\boxed{y_2 = 1 - \frac{1}{2}x\log\left(\frac{1+x}{1-x}\right)}$$

(4) Section 2.3 #22

$$9y'' + 6y' + 4y = 0, \quad y(0) = 3, \quad y'(0) = 4$$

c.e: $\quad 9r^2 + 6r + 4 = 0$

$$r = \frac{-6 \pm \sqrt{6^2 - 4\cdot 9\cdot 4}}{18} = -\frac{1}{3} \pm i\frac{\sqrt{3}}{3}.$$

then
$$y = e^{-\frac{1}{3}t}\left[A\cos\frac{\sqrt{3}}{3}t + B\sin\frac{\sqrt{3}}{3}t\right].$$

IC:
$$y(0) = 3 = A \Rightarrow A = 3.$$
$$y'(0) = 4 = -\frac{1}{3}A + \frac{\sqrt{3}}{3}B$$
$$\Rightarrow B = \frac{15}{\sqrt{3}}$$

$$\Rightarrow \boxed{y = e^{-\frac{1}{3}t}\left[3\cos\frac{\sqrt{3}}{3}t + \frac{15}{\sqrt{3}}\sin\frac{\sqrt{3}}{3}t\right]}$$

Section 2.3 # 27

$$y''' + 3y'' - 4y = 0$$

c.e: $r^3 + 3r^2 - 4 = 0$

Spot that $r = 1$ is a root so $r - 1$ is a factor of $r^3 + 3r^2 - 4$.

$$\begin{array}{r|rrrr} & 1 & 3 & 0 & -4 \\ -1 & & -1 & -4 & -4 \\ \hline & 1 & 4 & 4 & \end{array}$$

$$\Rightarrow r^3 + 3r^2 - 4 = (r-1)(r^2 + 4r + 4) = 0$$

$$r = \frac{-4 \pm \sqrt{16-16}}{2} = -2, -2$$

repeated roots.

so $\boxed{y = c_1 e^{t} + c_2 e^{-2t} + c_3 t e^{-3t}}$

Section 2.3 # 29

$$y''' + 27y = 0$$

C.E: $r^3 = -27$

$r^3 = 27e^{i\pi} \Rightarrow r_1 = 3e^{i\pi/3} = 3\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$r^3 = 27e^{i3\pi} \Rightarrow r_2 = 3e^{i\pi} = -3$

$r^3 = 27e^{i5\pi/3} \Rightarrow r_3 = 3e^{i5\pi/3} = 3e^{-i\pi/3}$

$= 3\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

so $\boxed{y = c_1 e^{-3t} + c_2 e^{\frac{3}{2}t}\left[A\cos\frac{3\sqrt{3}}{2}t + B\sin\frac{3\sqrt{3}}{2}t\right]}$

(5) $x'' + 4x = 0 \Rightarrow x = A\cos 2t + B\sin 2t.$

a) $x(0) = 1$, $x'(0) = -2$.

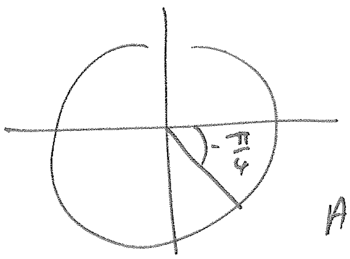
$x(0) = 1 = A \Rightarrow A = 1$

$x'(0) = -2 = 2B \Rightarrow B = -1.$

in form $x = C\cos(\omega t - \alpha)$,

$$C = \sqrt{A^2+B^2} = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\alpha = \tan^{-1}(B/A) = \tan^{-1}(-1) = -\frac{\pi}{4}$$



$A > 0,\ B < 0$

correct quadrant

so

$$x(t) = \sqrt{2}\cos(2t + \pi/4)$$

add 2π to $-\frac{\pi}{4}$

or

$$x(t) = \sqrt{2}\cos(2t - 7\pi/4)$$

b) $x(0) = -\frac{1}{2}, \quad x'(0) = \sqrt{3}$

$x(0) = A = \frac{-1}{2}, \quad x'(0) = 2B = \sqrt{3}$

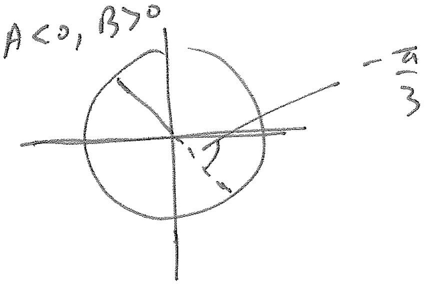
$\Rightarrow A = \frac{-1}{2}, \quad B = \frac{\sqrt{3}}{2}$

write in for

$x = C\cos(\omega t - \alpha)$

$$C = \sqrt{A^2+B^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

$$\tan^{-1}(B/A) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}.$$



need to add π:

$$\Rightarrow \alpha = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

$$\Rightarrow \boxed{x(t) = \cos\left(2t - \frac{2\pi}{3}\right)}$$

c) $x(0) = \frac{1}{2}, \quad x'(0) = \sqrt{3}.$

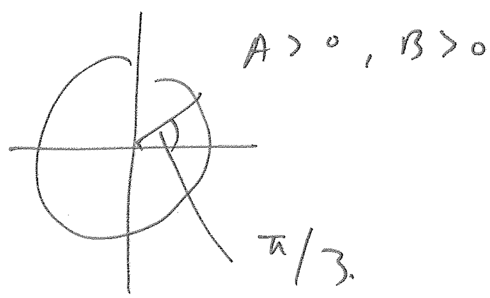
$x(0) = A = \frac{1}{2}, \quad x'(0) = 2B = \sqrt{3}.$

$\Rightarrow A = \frac{1}{2}, \quad B = \frac{\sqrt{3}}{2}.$

in form $C\cos(\omega t - \alpha)$:

$C = 1$ as before,

$\alpha = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$



$$\Rightarrow \boxed{x(t) = \cos(2t - \pi/3)}$$

d) $x(0) = -1, \quad x'(0) = -2$

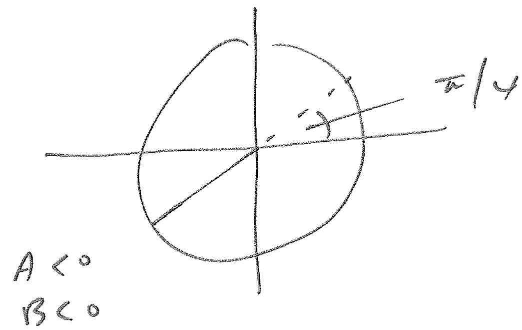
$x(0) = A = -1, \quad x'(0) = 2B = -2$

$\rightarrow \quad A = -1, \quad B = -1.$

in form $x = C\cos(\omega t - \alpha)$:

$C = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\alpha = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$



need to add π

$\rightarrow \quad \alpha = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

so $\boxed{x(t) = \sqrt{2}\cos\left(2t - \frac{5\pi}{4}\right)}$

(6) section 2.4 #17

$$mx'' + cx' + kx = 0$$

$m = 1$, $c = 8$, $k = 16$,

$x(0) = 5$, $x'(0) = -10$

$$x'' + 8x' + 16x = 0$$

C.E: $r^2 + 8r + 16 = 0$

$$r = \frac{-8 \pm \sqrt{64 - 64}}{2} = -4, -4$$

Critically damped

$$x(t) = (A + Bt)e^{-4t}.$$

$$x(0) = A = 5 \Rightarrow A = 5$$

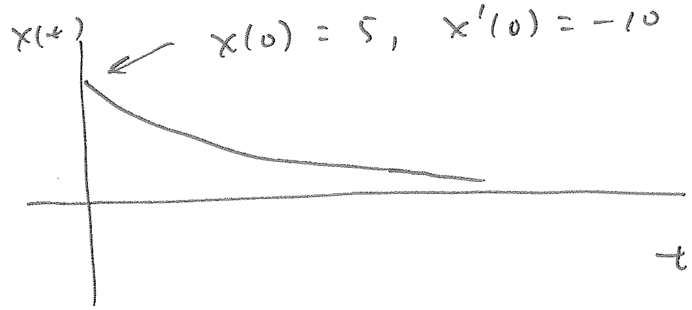
$$x'(t) = Be^{-4t} + (A + Bt)(-4)e^{-4t}.$$

$$x'(0) = B - 4A = -10$$

$$B = -10 + 4(5) = 10$$

then

$$x(t) = (5 + 10t)e^{-4t}$$



with $c = 0$,

$$u'' + 16u = 0, \Rightarrow r = \pm 4i$$

$$u(t) = A\cos 4t + B\sin 4t$$

$$u(0) = A = 5 \quad \Rightarrow \quad A = 5$$

$$u'(0) = 4B = -10 \quad \Rightarrow \quad B = \frac{-5}{2}$$

~~in form~~

in form $u = C\cos(\omega t - \alpha)$

$$C = \sqrt{5^2 + \frac{5^2}{2^2}} = \sqrt{25\left(1 + \frac{1}{4}\right)} = 5\sqrt{\frac{5}{4}}$$

$$= \frac{5}{2}\sqrt{5}$$

$$\alpha = \tan^{-1}(B/A) = \tan^{-1}\left(\frac{-1}{2}\right) \approx -.4636.$$

can also add 2π...

then

$$u(t) = \frac{5\sqrt{5}}{2}\cos(4t + .4636)$$