Homework #4 solutions

(1) Section 2.5 #4

$$4y'' + 4y' + y = 3xe^{x}$$

homog. $\Rightarrow$ C.E: $4r^2 + 4r + 1 = 0$

$$r = \frac{-4 \pm \sqrt{16-16}}{2 \cdot 8} = -\frac{1}{4}, -\frac{1}{4}.$$

so $y_1 = e^{-x/4}$, $y_2 = xe^{-x/4}$.

$y_p = (Ax+B)e^{x}$, $y_p' = Ae^{x} + (Ax+B)e^{x}$

$y_p'' = + 2Ae^{x} + (Ax+B)e^{x}$

then

$$4(Ax+B)e^{x} + 8Ae^{x} + 4Ae^{x} + 4(Ax+B)e^{x}$$
$$+ (Ax+B)e^{x} = 3xe^{x}$$

$$9(Ax+B)e^{x} + 12Ae^{x} = 3xe^{x}$$

coeff's : xe^x : $9A = 3 \Rightarrow A = 1/3$

e^x : $9B + 12A = 0$

$9B = -4 \Rightarrow B = \frac{-4}{9}$

So $\boxed{y_p = \left(\frac{1}{3}x - \frac{4}{9}\right)e^x}$

Section 2.5 # 5

$$y'' + y' + y = \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

neither are homog. terms

$$y_p = y_{p_1} + y_{p_2}$$

$$y_{p_1} = \frac{1}{2}$$

$$y_{p_2} = A\cos 2x + B\sin 2x$$

$$y_{p_2}' = -2A\sin 2x + 2B\cos 2x$$

$$y_{p_2}'' = -4A\cos 2x - 4B\sin 2x$$

then

$$[-3A + 2B]\cos 2x + [-3B - 2A]\sin 2x$$

$$= -\frac{1}{2}\cos 2x.$$

$\sin 2x$: $\quad -3B - 2A = 0 \quad \Rightarrow \quad A = -\frac{3}{2}B.$

$\cos 2x$: $\quad -3A + 2B = -\frac{1}{2}$

$$-3\left(-\frac{3}{2}\right)B + 2B = -\frac{1}{2}.$$

$$\frac{9B}{2} + 2B = -\frac{1}{2}.$$

$$\frac{13B}{2} = -\frac{1}{2} \quad \Rightarrow \quad B = -\frac{1}{13}.$$

then $\quad A = \frac{3}{2}\cdot\frac{1}{13} = \frac{3}{26}$

$$\Rightarrow y_{p_2} = \frac{3}{26}\cos 2x - \frac{1}{13}\sin 2x$$

$$\boxed{y_p = \frac{1}{2} + \frac{3}{26}\cos 2x - \frac{1}{13}\sin 2x}$$

Section 2.5 #8

$$y'' - 4y = \cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$$

C.E: $r^2 - 4 = 0 \quad r = \pm 2.$

then guess must have the form

$$y_p = \tilde{A} x e^{2x} + \tilde{B} x e^{-2x}$$

we can also write

$$y_p = A x \cosh 2x + B x \sinh 2x$$

$$= x\{ A \cosh 2x + B \sinh 2x]$$

$$y_p' = A \cosh 2x + B \sinh 2x + x [2A \sinh 2x + 2B \cosh 2x]$$

$$y_p'' = 2 [2A \sinh 2x + 2B \cosh 2x] + x [4A \cosh 2x + 4B \sinh 2x]$$

$$\Rightarrow \quad 4A \sinh 2x + 4B \cosh 2x = \cosh 2x$$

so

$B = \frac{1}{4}$, $A = 0$

$\Rightarrow$ $\boxed{y_p = \frac{1}{4} x \sinh 2x}$

Section 2.5 # 16

$y'' + 9y = 2x^2 e^{3x} + 5$

C.E: $r^2 + 9 = 0$ $\Rightarrow$ $r = \pm 3i$

so $\boxed{y_p = [Ax^2 + Bx + C] e^{3x} + D}$

Section 2.5 # 21

$y'' - 2y' + 2y = e^x \sin x$

C.E: $r^2 - 2r + 2 = 0$ $r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

so $y_1 = e^x \cos x, \quad y_2 = e^x \sin x.$

$$\Rightarrow \boxed{y_p = x e^x [A \cos x + B \sin x].}$$

Section 2.5 # 23

$$y'' + 4y = 3x \cos 2x$$

homog: $y_1 = \cos 2x, \quad y_2 = \sin 2x.$

$$\boxed{y_p = x\left[(Ax+B)\cos 2x + (Cx+D)\sin 2x\right]}$$

Section 2.5 # 25

$$y'' + 3y' + 2y = x\left(e^{-x} - e^{-2x}\right)$$

C.E: $(r+2)(r+1) = 0$

$r_1 = -2, \quad r_2 = -1.$

$y_1 = e^{-2x}, \quad y_2 = e^{-x}$

so $$\boxed{y_p = x(Ax+B)e^{-x} + x(Cx+D)e^{-2x}}$$

(2) $$y'' + y = \begin{cases} t & 0 \le t \le \pi \\ \pi e^{\pi - t} & , t > \pi \end{cases} \quad , y(0) = 0, \quad y'(0) = 1.$$

for $t \in [0, \pi]$,

$$y_l'' + y_l = t$$

$$y_{lh} = A\cos t + B\sin t.$$

$$y_{lp} = a_0 + a_1 t.$$

$$y_{lp}' = a_1, \quad y_{lp}'' = 0$$

$$\Rightarrow a_0 = 0, \quad a_1 = 1.$$

so $$y_l = A\cos t + B\sin t + t.$$

now IC's:

$y_l(0) = 0 \Rightarrow A = 0$

$y_l'(0) = 1 \Rightarrow B + 1 = 1 \Rightarrow B = 0$

so $\quad y_l(t) = t.$

for $t > \pi$:

$y_{rh} = C\cos t + D\sin t.$

$y_{rp} = be^{-t}, \quad y_{rp}' = -be^{-t},$

$y_{rp}'' = be^{-t}.$

then
$$be^{-t} + be^{-t} = \pi e^{\pi - t}.$$

$$\Rightarrow b = \frac{\pi}{2} e^{\pi}$$

then
$$y_r(t) = C\cos t + D\sin t + \frac{\pi}{2} e^{\pi - t}.$$

continuity at $t = \pi$ and differentiability

at $t = \pi$:

$y_r(\pi) = y_l(\pi), \quad y_r'(\pi) = y_l'(\pi)$

$y_r(\pi) = \pi = -C + \frac{\pi}{2} \Rightarrow C = -\frac{\pi}{2}.$

$y_r'(\pi) = 1 = -D - \frac{\pi}{2} \Rightarrow D = -1 - \frac{\pi}{2}.$

$\Rightarrow y_r = -\frac{\pi}{2}\cos t - \left(1+\frac{\pi}{2}\right)\sin t + \frac{\pi}{2}e^{\pi - t}$

finally,

$$y(t) = \begin{cases} t & 0 \le t \le \pi \\ -\frac{\pi}{2}\cos t - \left(1+\frac{\pi}{2}\right)\sin t + \frac{\pi}{2}e^{\pi-t} & t > \pi \end{cases}$$

[note, discontinuities in y or y' would have resulted in unboundedness ~~for~~ in y'']

(3) $y'' + y = \sin t$

first find that $y_1 = \cos t$, $y_2 = \sin t$.

$$W(t; y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= \cos^2 t + \sin^2 t = 1$$

then

$$\int \frac{-y_2 f(t)}{W} dt = \int \frac{-\sin t \sin t}{1} dt$$

$$= -\int \sin^2 t \, dt$$

$$= -\int \frac{1 - \cos 2t}{2} dt$$

$$= \int \frac{\cos 2t - 1}{2} dt$$

$$= \frac{1}{4} \sin 2t - \frac{t}{2}$$

$$\int y_1 \frac{f(t)}{W} dt = \int \cos t \sin t \, dt$$

$$= \int \frac{d}{dt} \frac{\sin^2 t}{2} \, dt$$

$$= \frac{\sin^2 t}{2} = \frac{1}{2}\left[\frac{1 - \cos 2t}{2}\right]$$

$$= \frac{1}{4} - \frac{1}{4} \cos 2t$$

can throw away wlog.

$$\Rightarrow y_p = -\frac{t}{2} \cos t + \frac{1}{4} \sin 2t \cos t - \frac{1}{4} \cos 2t \sin t$$

$$= -\frac{t}{2} \cos t + \frac{1}{4}\left[\sin 2t \cos t - \cos 2t \sin t\right]$$

$$= -\frac{t}{2} \cos t + \frac{1}{4} \sin(2t - t)$$

$$= -\frac{t}{2} \cos t + \frac{1}{4} \sin t$$

homog. term

$$\Rightarrow y_p = -\frac{t}{2} \cos t.$$