Homework 5 solutions

1 section 4.1 #1

$$\mathcal{L}(t) = \int_0^\infty e^{-st}\, t\, dt = \cancel{-\frac{t}{s} e^{-st} \Big|_0^\infty} + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= -\frac{1}{s^2} e^{-st} \Big|_0^\infty = \frac{1}{s^2}, \quad s > 0$$

$$\boxed{\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0}$$

section 4.1 #3

$$\mathcal{L}(e^{3t+1}) = e\,\mathcal{L}(e^{3t}) = e\int_0^\infty e^{-st} e^{3t}\, dt$$

$$= e\int_0^\infty e^{-(s-3)t}\, dt = -\frac{e}{s-3} e^{-(s-3)t} \Big|_0^\infty$$

$$= \frac{e}{s-3}, \quad s > 3.$$

$$\boxed{\mathcal{L}(e^{3t+1}) = \frac{e}{s-3}, \quad s > 3}$$

#6 Section 4.1 #6

$$\mathcal{L}(\sin^2 t) = \int_0^\infty e^{-st} \sin^2 t \, dt$$

$$= \int_0^\infty e^{-st} \left(\frac{1-\cos 2t}{2} \right) dt$$

$$= \int_0^\infty e^{-st} \frac{1}{2} dt - \frac{1}{2} \int_0^\infty e^{-st} \cos 2t \, dt$$

$$= -\frac{1}{2s} e^{-st} \Big|_0^\infty - \frac{1}{2} \int_0^\infty e^{-st} \left(\frac{e^{i2t} + e^{-i2t}}{2} \right) dt$$

$$= \frac{1}{2s} - \frac{1}{4} \left[\int_0^\infty e^{-st} e^{i2t} dt + \int_0^\infty e^{-st} e^{-i2t} dt \right]$$

$$= \frac{1}{2s} - \frac{1}{4} \left[\int_0^\infty e^{-(s-2i)t} dt + \int_0^\infty e^{-(s+2i)t} dt \right]$$

$$= \frac{1}{2s} - \frac{1}{4} \left[\frac{-1}{s-2i} e^{-(s-2i)t} \Big|_0^\infty - \frac{1}{s+2i} e^{-(s+2i)t} \Big|_0^\infty \right]$$

$$= \frac{1}{2s} - \frac{1}{4} \left[\frac{1}{s-2i} + \frac{1}{s+2i} \right] = \frac{1}{2s} - \frac{1}{4} \left[\frac{2s}{s^2+4} \right].$$

$$= \frac{1}{2s} + \frac{1}{2}\,\frac{s}{s^2+4}$$

$$\Rightarrow \boxed{\mathcal{L}(\sin^2 t) = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right], \quad s > 0}$$

section 4.1 # 10

$f(t)$

$$\mathcal{L}(f(t)) = ?$$

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^1 (1-t)\,e^{-st}\,dt$$

$$= \int_0^1 e^{-st}\,dt - \int_0^1 t e^{-st}\,dt$$

$$= \frac{-1}{s} e^{-st}\Big|_0^1 - \left[\frac{-t}{s} e^{-st}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt\right]$$

$$= \frac{-1}{s}\left[e^{-s} - 1\right] - \left[\frac{-1}{s} e^{-s} - \frac{1}{s^2} e^{-st}\Big|_0^1\right]$$

$$= \frac{1}{s} + \frac{1}{s^2}\left(e^{-s} - 1\right) = \frac{1}{s^2}\left(e^{-s} + s - 1\right)$$

$$\Rightarrow \boxed{\mathcal{L}(f(t)) = \frac{e^{-s}+s-1}{s^2}}$$

(2) Section 4.1 #23

$$\mathcal{L}^{-1}\left(\frac{3}{s^4}\right) = 3\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = 3\mathcal{L}^{-1}\left(\frac{1}{s^{3+1}}\right)$$

$$= \frac{3}{3!}\mathcal{L}^{-1}\left(\frac{3!}{s^{3+1}}\right) = \frac{3}{6}t^3$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{3}{s^4}\right) = \frac{1}{2}t^3}$$

Section 4.1 #25

$$\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{2}{s^{5/2}}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - 2\mathcal{L}^{-1}\left(\frac{2}{s^{5/2}}\right)$$

$$= 1 - 2\mathcal{L}^{-1}\left(\frac{1}{s^{3/2+1}}\right) = 1 - \frac{2}{\Gamma(\frac{3}{2}+1)}\mathcal{L}^{-1}\left(\frac{\Gamma(\frac{3}{2}+1)}{s^{3/2+1}}\right)$$

$$= 1 - \frac{2}{\Gamma(3/2+1)}t^{3/2} = 1 - \frac{2}{\frac{3}{2}\Gamma(\frac{3}{2})}t^{3/2}$$

$$= 1 - \frac{4}{3\Gamma(\frac{1}{2}+1)} t^{3/2} = 1 - \frac{4}{3 \cdot \frac{1}{2}\Gamma(\frac{1}{2})} t^{3/2}$$

$$= 1 - \frac{8}{3\sqrt{\pi}} t^{3/2}$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{2}{s^{5/2}}\right) = 1 - \frac{8}{3\sqrt{\pi}} t^{3/2}}$$

Section 4.1 # 29

$$F(s) = \frac{5 - 3s}{s^2+9} = 5\frac{1}{s^2+9} - 3\frac{s}{s^2+9}$$

$$= \frac{5}{3}\frac{3}{s^2+3^2} - 3\frac{s}{s^2+9}$$

$$\boxed{\mathcal{L}^{-1}(F) = \frac{5}{3}\sin 3t - 3\cos 3t}$$

section 4.1 # 31

$$F(s) = \frac{10s - 3}{25 - s^2} = \frac{3 - 10s}{s^2 - 25}$$

$$= \frac{3}{s^2-25} - 10\frac{s}{s^2-25}$$

$$= \frac{3}{5}\frac{5}{s^2-25} - 10\frac{s}{s^2-25}$$

$$\boxed{\mathcal{L}^{-1}(F(s)) = \frac{3}{5}\sinh 5t - 10\cosh 5t}$$

(3) <u>Section 4.2 #2</u>

$$x'' + 9x = 0 \qquad x(0) = 3, \; x'(0) = 4.$$

apply $\mathcal{L}(\cdot)$:

$$s^2 X(s) - s\,x(0) - x'(0) + 9X(s) = 0.$$

$$(s^2+9)X(s) = 3s+4$$

$$X = \frac{3s+4}{s^2+9} = 3\frac{s}{s^2+3^2} + \frac{4}{3}\frac{3}{s^2+3^2}$$

$$\boxed{x(t) = 3\cos 3t + \frac{4}{3}\sin 3t}$$

Section 4.2 # 4

$$x'' + 8x' + 15x = 0, \quad x(0) = 2, \quad x'(0) = -3$$

apply $\mathcal{L}(\cdot)$:

$$s^2 X - s x(0) - x'(0) + 8(sX - x(0)) + 15X = 0.$$

$$(s^2 + 8s + 15) X(s) = 2s - 3 + 16$$

$$X(s) = \frac{2s+13}{s^2+8s+15} = \frac{2s+13}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

$$2s + 13 = A(s+5) + B(s+3)$$

$\underline{s = -3}$: $7 = 2A \Rightarrow A = 7/2$

$s = -5$: $3 = -2B \Rightarrow B = -3/2$

$$X(s) = \frac{7/2}{s+3} - \frac{3/2}{s+5}$$

$$\boxed{x(t) = \frac{7}{2} e^{-3t} - \frac{3}{2} e^{-5t}}$$

section 4.2 #8

$$x'' + 9x = 1, \qquad x(0) = x'(0) = 0$$

$$(s^2+9)X(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$$1 = A(s^2+9) + s(Bs+C)$$

$s=0$: $\quad 1 = 9A \Rightarrow A = 1/9$

$s=3i$: $\quad 1 = 3i(3iB+C) \Rightarrow C=0,\ B=-1/9$

so

$$X(s) = \frac{1/9}{s} + \frac{(-1/9)s}{s^2+9}$$

$$\boxed{x(t) = \frac{1}{9} - \frac{1}{9}\cos 3t}$$

Section 4.2 # 11

$$x' = 2x + y \qquad x(0) = 1$$
$$y' = 6x + 3y \qquad y(0) = -2$$

$$sX(s) - x(0) = 2X + Y$$

$$sY(s) - y(0) = 6X + 3Y$$

$$(s-2)X - Y = 1$$

$$-6X + (s-3)Y = -2$$

$$\begin{pmatrix} s-2 & -1 \\ -6 & s-3 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$X(s) = \frac{\begin{vmatrix} 1 & -1 \\ -2 & s-3 \end{vmatrix}}{\begin{vmatrix} s-2 & -1 \\ -6 & s-3 \end{vmatrix}} = \frac{s-3-2}{s^2-5s+6-6}$$

$$= \frac{s-5}{s^2-5s} = \frac{s-5}{s(s-5)} = \frac{1}{s}$$

$$\Rightarrow \quad X(s) = \frac{1}{s}$$

$$\Rightarrow \boxed{x(t) = 1}$$

$$Y(s) = \frac{\begin{vmatrix} s-2 & 1 \\ -6 & -2 \end{vmatrix}}{s(s-5)} = \frac{-2(s-2)+6}{s(s-5)}$$

$$= \frac{-2s+10}{s(s-5)} = \frac{-2(s-5)}{s(s-5)} = \frac{-2}{s}$$

$$\Rightarrow \boxed{y(t) = -2}$$

5 Section 4.3 # 3

$$f(t) = e^{-2t} \sin 3\pi t.$$

$$\mathcal{L}(\sin 3\pi t) = \frac{3\pi}{s^2 + 9\pi^2}$$

then

$$\boxed{\mathcal{L}(e^{-2t} \sin 3\pi t) = \frac{3\pi}{(s+2)^2 + 9\pi^2}}$$

Section 4.3 #4

$$f(t) = e^{-t/2}\cos(2t - \pi/4)$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

so

$$\cos(2t - \pi/4) = \cos 2t \cos\frac{\pi}{4} + \sin 2t \sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}\left[\cos 2t + \sin 2t\right].$$

$$\mathcal{L}(\cos 2t) = \frac{s}{s^2+4}, \quad \mathcal{L}(\sin 2t) = \frac{2}{s^2+4}$$

so

$$\mathcal{L}^{-1}(f) = \frac{1}{\sqrt{2}}\left[\frac{s+1/2}{(s+1/2)^2+4} + \frac{2}{(s+1/2)^2+4}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{s+1/2+2}{s^2+s+1/4+4}\right] = \frac{1}{\sqrt{2}}\left[\frac{s+5/2}{s^2+s+17/4}\right]$$

$$= \frac{\sqrt{2}}{2}\left[\frac{s+5/2}{s^2+s+17/4}\right] = \sqrt{2}\left[\frac{2s+5}{4s^2+4s+17}\right]$$

so

$$\boxed{\mathcal{L}^{-1}\left(e^{-t/2}\sin\left(2t-\pi/4\right)\right) = \sqrt{2}\left[\frac{2s+5}{4s^2+4s+17}\right]}$$

section 4.3 #6

$$F(s) = \frac{s-1}{(s+1)^3} = \frac{s+1-2}{(s+1)^3}$$

$$= \frac{s+1}{(s+1)^3} - \frac{2}{(s+1)^3}$$

$$= \frac{1}{(s+1)^2} - \frac{2}{(s+1)^3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}t.$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{2}{s^3}\right) = \frac{1}{2}t^2$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+1)^3}\right) = \frac{1}{2}e^{-t}t^2$$

$$\Rightarrow \mathcal{L}^{-1}(F) = te^{-t} - 2\cdot\frac{1}{2}t^2e^{-t}$$

$$= e^{-t}(t-t^2)$$

5

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = \sin 2t.$$

a) let $u'' = \phi(t)$

then

$$\int_0^t (t-\xi)\phi(\xi)\,d\xi = \int_0^t (t-\xi)u''\,d\xi$$

$$= \int_0^t t u''\,d\xi - \int_0^t \xi u''\,d\xi$$

$$= t u'\Big|_0^t - \left[\xi u'\Big|_0^t - \int_0^t u'\,d\xi\right]$$

$$\cancel{= tu'(t) - [}$$

$$= t\left[\cancel{u'(t)} - u'(0)\right] - \left[\cancel{tu'(t)} - u(t) + u(0)\right]$$

$$= -u'(0)t + u(t) - u(0)$$

then we have

$$u'' + u(t) - u'(0)\,t - u(0) = \sin 2t$$

b) let $u = u_{p_1}(t) + u_{p_2}(t)$

where

$$u_1'' + u_1 = \sin 2t$$

$$u_2'' + u_2 = u'(0)\,t + u(0)$$

so

$$u_2 = At + B$$

therefore,

$$u = u_1(t) + At + B$$

$$u'' = u_1'' + 0 = \phi$$

so in fact u_2 doesn't factor into the solution for ϕ

so we may take $A = B = 0$ wlog,

or equivalently $u'(0) = u(0) = 0$

b) so we have

$$u'' + u = \sin 2t \qquad u(0) = u'(0) = 0$$

$$u = u_h + u_p$$

$$u_h = A\cos t + B\sin t$$

$$u_p = C\sin 2t + D\cos 2t \qquad (D = 0)$$

so

$$-4C\sin 2t + C\sin 2t = \sin 2t.$$

$$-3C = 1 \Rightarrow C = \frac{-1}{3}.$$

so

$$u = A\cos t + B\sin t - \frac{1}{3}\sin 2t$$

$$u(0) = 0 \Rightarrow A = 0$$

$$u'(0) = B - \frac{2}{3} \Rightarrow B = \frac{2}{3}$$

$$\Rightarrow \boxed{u(t) = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t}$$

c) $\phi(t) + \int_0^t (t-\xi)\,\phi(\xi)\,d\xi = \sin 2t$

take $\mathcal{L}(\cdot)$: $\quad \phi(t) \to \Phi(s)$

$$\Phi(s) + \frac{1}{s^2}\Phi = \frac{2}{s^2+4}$$

$$\frac{s^2\Phi + \Phi}{s^2} = \frac{2}{s^2+4}$$

$$\Phi = \frac{2s^2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$2s^2 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$\underline{s=i}$ $\quad -2 = (iC+D)(3) \Rightarrow C=0,\ D=\frac{-2}{3}$

$\underline{s=2i}$ $\quad -8 = (A\cdot 2i+B)(-3) \Rightarrow A=0,\ B=\frac{8}{3}$

so

$$\Phi(s) = \frac{8}{3}\frac{1}{s^2+4} - \frac{2}{3}\frac{1}{s^2+1}$$

$$= \frac{4}{3}\frac{2}{s^2+4} - \frac{2}{3}\frac{1}{s^2+1}$$

$$\boxed{\phi(t) = \frac{4}{3}\sin 2t - \frac{2}{3}\sin t}$$

recall $u(t) = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t$

$$u''(t) = -\frac{2}{3}\sin t + \frac{4}{3}\sin 2t = \phi(t)$$

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