Homework 6 solutions

(1) Section 4.4 # 10

$$F(s) = \frac{1}{s^2(s^2+k^2)}$$

convolution thm:

$$\mathcal{L}^{-1}\big(G(s)H(s)\big) = \int_0^t g(\tau)h(t-\tau)\,d\tau$$

let $H(s) = \frac{1}{s^2} \Rightarrow h(t) = t$

$G(s) = \frac{1}{s^2+k^2} \Rightarrow g(t) = \frac{1}{k}\sin kt$

$$\mathcal{L}^{-1}(F(s)) = f(t) = \int_0^t \left(\frac{1}{k}\sin k\tau\right)(t-\tau)\,d\tau$$

$$= \frac{1}{k}\left[\int_0^t t\sin k\tau\,d\tau - \int_0^t \tau\sin k\tau\,d\tau\right]$$

now

$$\int_0^t t\sin k\tau\,d\tau = t\int_0^t \sin k\tau\,d\tau$$

$$= t\left[-\frac{1}{k}\cos k\tau\right]_0^t = t\left[-\frac{1}{k}\cos kt + \frac{1}{k}\right]$$

and

$$\int_0^t \tau \sin k\tau \, d\tau = -\tau \frac{1}{k}\cos k\tau \Big|_0^t + \int_0^t \frac{1}{k}\cos k\tau \, d\tau$$

$$= -\frac{t}{k}\cos kt + 0 + \frac{1}{k^2}\sin k\tau \Big|_0^t$$

$$= -\frac{t}{k}\cos kt + \frac{1}{k^2}\sin kt$$

so

$$\int_0^t t \sin k\tau \, d\tau - \int_0^t \tau \sin k\tau \, d\tau = \frac{1}{k}t - \frac{1}{k^2}\sin kt$$

$$\Rightarrow \boxed{f(t) = \frac{1}{k^3}\left[kt - \sin kt\right]}$$

Section 4.4 #21

$$f(t) = \frac{e^{3t}-1}{t}$$

first,

$$f(t) \sim \frac{1+3t-1}{t} \sim 3 \quad \text{as } t \to 0$$

so Laplace transform exists.

use

$$\mathcal{L}\left(g(t)/t \right) = \int_s^\infty G(\sigma)\, d\sigma$$

where $G(s) = \mathcal{L}(g(t))$. ; $g(t) = e^{3t} - 1$.

then

$$G(s) = \frac{1}{s-3} - \frac{1}{s}$$

$$\Rightarrow \mathcal{L}(f(t)) = \int_s^\infty \frac{1}{\sigma-3} - \frac{1}{\sigma}\, d\sigma$$

$$= \log(\sigma-3) - \log\sigma \Big|_s^\infty$$

$$= \log s - \log(s-3)$$

(the terms at ∞ cancel).

$$\rightarrow \boxed{\mathcal{L}(f(t)) = \log \frac{s}{s-3}}$$

* if $f(t) = \dfrac{e^{3t}}{t}$, $f(t) \sim \dfrac{1}{t}$ as $t \to \infty$

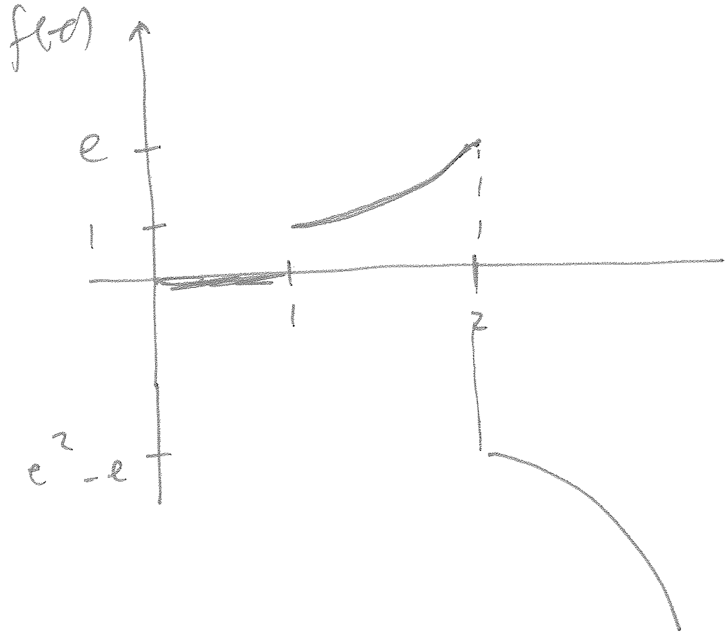
~~this~~ is then $\int_0^\infty e^{-st} f(t)\,dt$ would not be integrable near $t=0$. The -1 is therefore critical in ensuring that the transform exist.

② Section 4.5 # 4

$$F(s) = \frac{e^{-s} - e^{2-2s}}{s-1} = e^{-s}\frac{1}{s-1} - e^{2}e^{-2s}\frac{1}{s-1}$$

$$f(t) = u(t-1)e^{(t-1)} - e^{2}\,u(t-2)e^{(t-2)}$$

$$\boxed{f(t) = u(t-1)\,e^{t-1} - u(t-2)\,e^{t}}$$



<u>section 4.5 #21</u>

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 2-t & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$

$$f(t) = t + u(t-1)(2-t-t) + u(t-2)\left(-t \overset{-}{\cancel{+}} 2 + 2 - t\right)$$

$$= t \overset{-}{\cancel{+}} 2u(t-1)(t-1) + u(t-2)(t-2).$$

$$\Rightarrow \boxed{\mathcal{L}(f(t)) = \frac{1}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right].}$$

(3) Section 4.6 #21b

$$i'' + 60i' + 1000i = 10\sum_{n=0}^{\infty}(-1)^n\,\delta\left(t - \frac{n\pi}{10}\right)$$

$$i(0) = i'(0) = 0$$

$\mathcal{L}(\cdot)$:

$$(s^2 + 60s + 1000)\,I(s) = 10\sum_{n=0}^{\infty}(-1)^n e^{-n\frac{\pi}{10}s}$$

$$I(s) = 10\sum_{n=0}^{\infty}(-1)^n \frac{e^{-n\frac{\pi}{10}s}}{s^2 + 60s + 900 + 100}$$

$$\mathcal{L}^{-1}\left(\frac{e^{-n\frac{\pi}{10}s}\cdot 10}{(s+30)^2 + 10^2}\right) = u(t - n\pi/10)\,\mathcal{L}^{-1}\left(\frac{10}{(s+30)^2 + 10^2}\right)$$

$$= u(t - n\pi/10)\, e^{-30\left(t - \frac{n\pi}{10}\right)} \sin\left(10\left(t - \frac{n\pi}{10}\right)\right)$$

$$= (-1)^n e^{-30t} \sin(10t)\, e^{3n\pi}\, u(t - n\pi/10)$$

$$\Rightarrow i(t) = e^{-30t} \sin 10t \sum_{n=0}^{\infty} u(t - n\pi/10)\, e^{3n\pi}$$

on interval $\frac{k\pi}{10} < t < \frac{(k+1)\pi}{10}$,

only $u(t - n\pi/10)$, $n = 0, 1, 2, \ldots, k$ contribute to the sum.

So on this interval,

$$i(t) = e^{-30t} \sin t \sum_{n=0}^{k} \left(e^{3\pi}\right)^n$$

Now let

$$S' = 1 + z + \cdots + z^k$$

$$-\; zS' = z + z^2 + \cdots + z^{k+1}$$

$$(1-z)\, S' = 1 - z^{k+1}$$

$$S' = \frac{z^{k+1} - 1}{z - 1}$$

$\Rightarrow$ with $z = e^{3\pi}$,

$$\boxed{i(t) = e^{-30t} \sin t \left[\frac{e^{3\pi(n+1)} - 1}{e^{3\pi} - 1} \right].}$$ on $\frac{k\pi}{10} < t < \frac{(k+1)\pi}{10}$

Section 4.6 # 22

$$x'' + x = \sum_{n=0}^{\infty} \delta(t - 2n\pi) \qquad x(0) = x'(0) = 0.$$

$$(s^2+1)\, X(s) = \sum_{n=0}^{\infty} e^{-2n\pi s}$$

$$X(s) = \sum_{n=0}^{\infty} \frac{e^{-2n\pi s}}{s^2+1}$$

$$x(t) = \sum_{n=0}^{\infty} u(t-2n\pi)\sin(t-2n\pi)$$

$$= \sum_{n=0}^{\infty} u(t-2n\pi)\sin t.$$

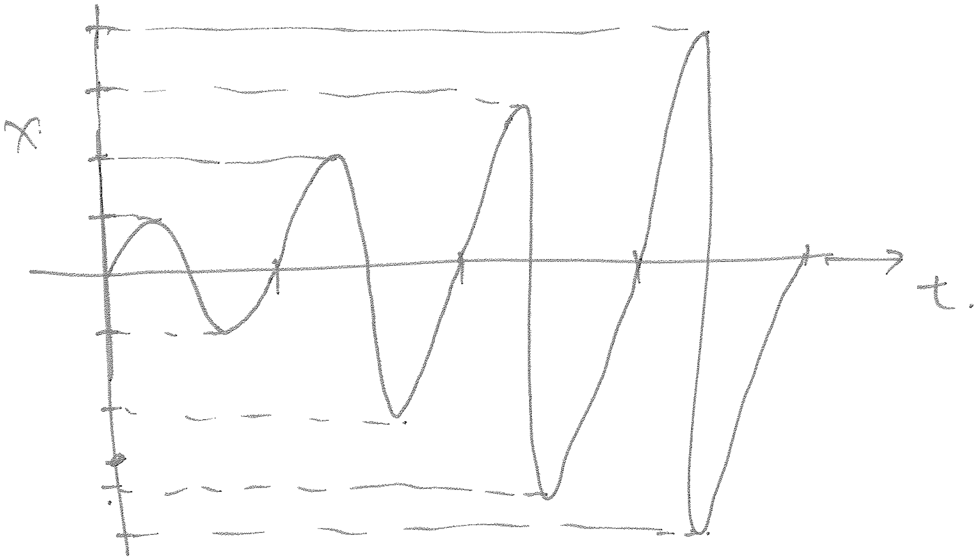
When $2k\pi < t < 2(k+1)\pi$,

sum only goes up to k since $u(t-2n\pi) = 0$

when ~~the~~ $n \geq 2(k+1)\pi$

$\Rightarrow$ on this interval,

$$x(t) = \sin t \sum_{n=0}^{k} 1 = (k+1)\sin t.$$



(4) Section 5.1 # 2

$$x^{(iv)} + 6x'' - 3x' + x = \cos 3t.$$

$$x^{(iv)} = \cos 3t - 6x'' + 3x' - x$$

let

$$x = x_1$$

$$x_1' = x_2$$

$$x_2' = x_3 = x_1''$$

$$x_3' = x_4 = x_1'''$$

$$x_4' = \cos 3t - 6x_3 + 3x_2 - x_1$$

$$\boxed{\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos 3t \end{pmatrix}}$$

5) Section 5.4 #4

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

find eve's evr's:

$$\det\begin{pmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{pmatrix} = 0$$

$$(4-\lambda)(-1-\lambda) - 6 = 0$$

$$(\lambda+1)(\lambda-4) - 6 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda-5)(\lambda-2) = 0$$

$$\lambda_+ = 5, \quad \lambda_- = -2.$$

$\vec{\xi}^{\,+}$:
$$\begin{pmatrix} 4-5 & 1 \\ \underline{\quad} & \underline{\quad} \end{pmatrix}\begin{pmatrix} \xi_1^+ \\ \xi_2^+ \end{pmatrix} = 0$$

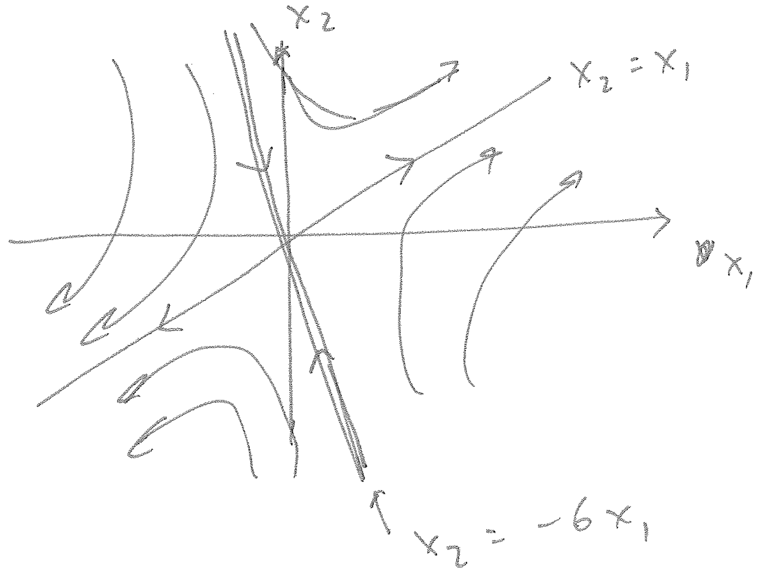
$$-\xi_1^+ + \xi_2^+ = 0 \Rightarrow \vec{\xi}^{\,+} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\vec{\xi}^{\,-}$:
$$\begin{pmatrix} 4-(-2) & 1 \\ \underline{\quad} & \underline{\quad} \end{pmatrix}\begin{pmatrix} \xi_1^- \\ \xi_2^- \end{pmatrix} = 0$$

$$6\xi_1^- + \xi_2^- = 0 \Rightarrow \vec{\xi}^- = \begin{pmatrix} 1 \\ -6 \end{pmatrix}.$$

then

$$\vec{x} = c_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + c_- \begin{pmatrix} 1 \\ -6 \end{pmatrix} e^{-2t}.$$



Saddle point, unstable.

Section 5.4 #6

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\det \begin{pmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{pmatrix} = 0 \Rightarrow (9-\lambda)(-2-\lambda) + 30 = 0$$

$$\lambda^2 - 7\lambda - 18 + 30 = 0$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda - 3)(\lambda - 4) = 0$$

$\lambda_+ = 4, \quad \lambda_- = 3$

$\vec{\xi}^{+}$:

$$\begin{pmatrix} 9-4 & 5 \\ - & - \end{pmatrix} \begin{pmatrix} \xi_1^+ \\ \xi_2^+ \end{pmatrix} = 0$$

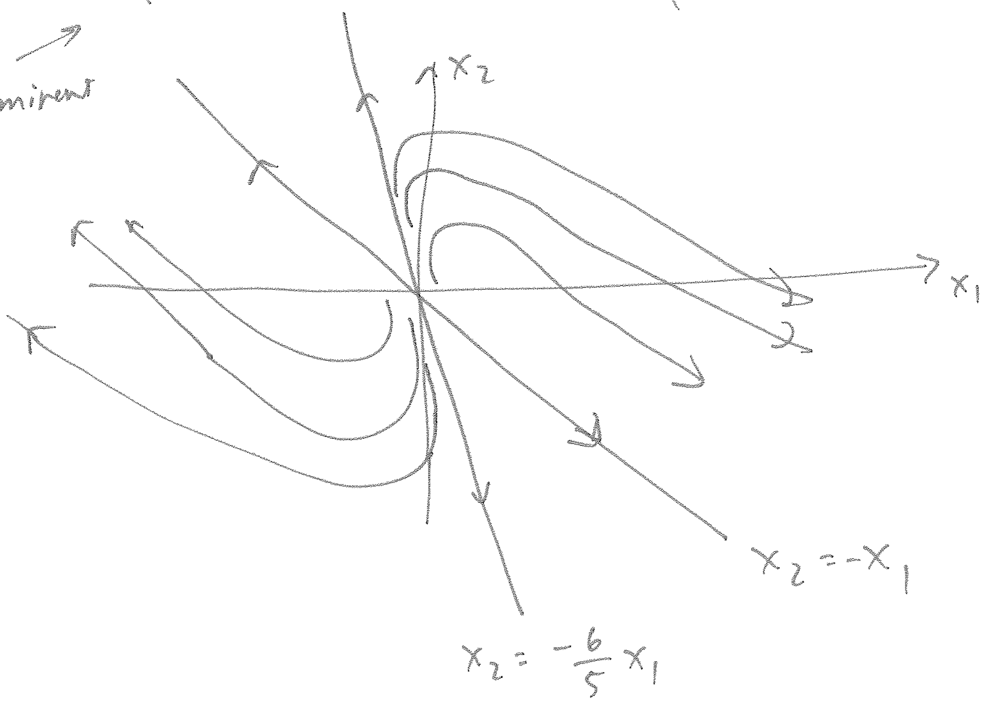
$$5\xi_1^+ + 5\xi_2^+ = 0 \Rightarrow \vec{\xi}^+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$\vec{\xi}^{-}$: $$\begin{pmatrix} 9-3 & 5 \\ - & - \end{pmatrix} \begin{pmatrix} \xi_1^- \\ \xi_2^- \end{pmatrix} = 0.$$

$$6\xi_1^- + 5\xi_2^- = 0 \Rightarrow \vec{\xi}^- = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$$

$$\vec{x} = c_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_- \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}.$$

dominant



unstable, node.

Section 5.4 # 8

in class

Section 5.4 #11

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{1}{-6+5} \begin{pmatrix} -6 & -5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= -1 \begin{pmatrix} -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

$$\Rightarrow \vec{x}(t) = 6 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} - \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}$$

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

$$\det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = 0 \Rightarrow (\lambda - 1)^2 + 4 = 0$$

so $\lambda_\pm = 1 \pm 2i$

$$\vec{\xi}^+ : \quad \begin{pmatrix} 1-(1+2i) & -2 \\ - & - \end{pmatrix} \begin{pmatrix} \xi_1^+ \\ \xi_2^+ \end{pmatrix} = 0$$

$$\begin{pmatrix} -2i & -2 \\ - & - \end{pmatrix} \begin{pmatrix} \xi_1^+ \\ \xi_2^+ \end{pmatrix} = 0$$

$$-2i\,\xi_1^+ - 2\,\xi_2^+ = 0 \Rightarrow \vec{\xi}^+ = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

$$\vec{\xi}^+ e^{\lambda_+ t} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(1+2i)t} = e^t \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i2t}$$

$$= e^t \begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos 2t + i \sin 2t)$$

$$= e^t \begin{pmatrix} \cos 2t + i \sin 2t \\ \sin 2t - i \cos 2t \end{pmatrix}$$

$$= e^t \left[\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ -\cos 2t \end{pmatrix} \right]$$

$$\Rightarrow \vec{x} = c_1 \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} e^t + c_2 \begin{pmatrix} \sin 2t \\ -\cos 2t \end{pmatrix} e^t$$

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

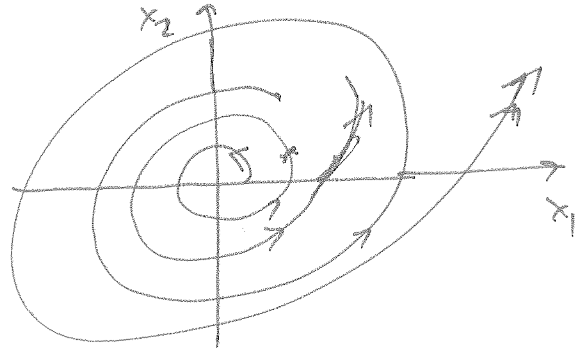
$$\Rightarrow c_1 = 0, \quad c_2 = -4$$

so

$$\vec{x} = \begin{pmatrix} -4 \sin 2t \\ 4 \cos 2t \end{pmatrix} e^t$$

~~direction~~
direction:

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



unstable, spiral.

6

$$x'' + x = [u(t-2\pi) - u(t-4\pi)] \sin t \equiv f(t)$$

$x(0) = x'(0) = 0.$

$$x'' + x = u(t-2\pi)\sin(t-2\pi) - u(t-4\pi)\sin(t-4\pi)$$

a) $$\mathcal{L}(f) = e^{-2\pi s}\frac{1}{s^2+1} - e^{-4\pi s}\frac{1}{s^2+1}$$

b) $$(s^2+1)X(s) = e^{-2\pi s}\frac{1}{(s^2+1)} - e^{-4\pi s}\frac{1}{(s^2+1)}$$

$$X(s) = e^{-2\pi s}\frac{1}{(s^2+1)^2} - e^{-4\pi s}\frac{1}{(s^2+1)^2}$$

now

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \frac{1}{2}(\sin t - t\cos t)$$

$$\Rightarrow x(t) = \frac{1}{2}u(t-2\pi)\left[\sin(t-2\pi) - (t-2\pi)\cos(t-2\pi)\right]$$

$$- \frac{1}{2}u(t-4\pi)\left[\sin(t-4\pi) - (t-4\pi)\cos(t-4\pi)\right]$$

$$= \frac{1}{2}u(t-2\pi)\left[\sin t - (t-2\pi)\cos t\right] - \frac{1}{2}u(t-4\pi)\left[\sin t - (t-4\pi)\cos t\right]$$

c) on $0 < t < 2\pi$,

$$x'' + x = 0, \quad x(0) = x'(0) = 0$$

so $x = 0$ on this interval

on $2\pi < t < 4\pi$,

$$x'' + x = \sin t, \quad x(2\pi) = x'(2\pi) = 0$$

there is resonant behavior on this interval.

on ~~$4\pi < t <$~~ $t > 4\pi$,

obtain from solution on interval $2\pi < t < 4\pi$

$$x'' + x = 0, \quad x(4\pi) = x(4\pi^-)$$
$$x'(4\pi) = x'(4\pi^-)$$

there is resonance only on the interval $2\pi < t < 4\pi$, so amplitude does <u>not</u> go to ∞. as $t \to \infty$

d) for $t > 4\pi$,

$$x(t) = \pi \cos t - 2\pi \cos t = -\pi \cos t$$

amplitude $= \underline{\underline{\pi}}$